

Cardinality:

A and B are the same size iff there exists a bijection $A \leftrightarrow B$

A is "smaller than" B if $\exists A \hookrightarrow B$ but no $B \hookrightarrow A$

Ordinality, or Order Type

$(P, <)$ P a set, $<$ a binary relation

Poset:

- < partially orders P if
 - $x \neq x$ (antireflexive)
 - If $x < y$ then $y \neq x$ (antisymmetric)
 - If $x < y$ and $y < z$ then $x < z$

Toset:

- < totally orders P if, in addition,
 ~~$\forall x, y \in P, x < y$~~
 - $\forall x, y \in P, x < y \text{ or } x > y \text{ or } x = y$

Woset:

- < well orders P if, in addition,
 - $\forall S \subseteq P \setminus \{S \neq \emptyset\}$, Aaugh, not enough room!

Woset:

< well-orders P if, in addition,
 \forall nonempty $S \subseteq P$, $\exists x \in S$ st. $\forall y \in S$, $x \leq y$

That is, any nonempty subset of P has a least element.

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Why Wosets?

Because they're where induction lives!

Let $\phi(x)$ stand for the statement:

"The fact ϕ is true about x ."

Wosets satisfy the Least Member Principle

If there is a $w \in W$ with $\phi(w)$, then
there is a least $w \in W$ with $\phi(w)$.

This gives us the Principle of Induction:

To prove that $\phi(x)$ is true for all $x \in \omega$,
all we have to do is show that

- If $\phi(y)$ is true for all $y < x$ then $\phi(x)$ is true.

Why does this work?

Otherwise take smallest x with $\neg\phi(x)$.

True that $\phi(y)$ ~~for~~ holds for all $y < x$

so must have $\phi(x) \Rightarrow \Leftarrow$

How do wosets give us a notion of size?

Let (P, \lessdot) and (Q, \triangleleft) be posets.

$$f: P \rightarrow Q$$

is an order-preserving map if $\forall a, b \in P$

$$a \lessdot b \Rightarrow f(a) \triangleleft f(b)$$

Fact: If W is a ^(totet) woset, and $f: W \rightarrow Q$ op.
then f is injective.

Proof:

Let $a \neq b \in W$. Then either $a \lessdot b$ or $a \gtrdot b$.

Thus either $f(a) \triangleleft f(b)$ or $f(b) \triangleleft f(a)$.

In particular, $f(a) \neq f(b)$. \square

Fact: If W is a ^(totet) woset and $f: W \rightarrow Q$ o.p. then
 $f(a) \triangleleft f(b) \Rightarrow a \lessdot b$.

Proof:

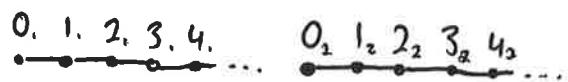
Suppose $f(a) \triangleleft f(b)$. Then $a \neq b$, and $a \neq b$,
so must have $a \lessdot b$.

Given posets (W, \leq) and (U, \trianglelefteq) we say W and U are the "same size" if there is an o.p. bijection $f: W \rightarrow U$.

This is strictly stronger than our cardinality criterion!

Consider:

$(\mathbb{N}, <)$ and $(\mathbb{N} \times \mathbb{N}) \setminus \{(n, n) \mid n \in \mathbb{N}\} \cup \{\infty\}$



Are there bijections? Sure! Lots!

Is there an order-preserving bijection?

Suppose there were. $f: \mathbb{N} \leftrightarrow \mathbb{N}_1 \cup \mathbb{N}_2$ o.p.

Then $\exists n \in \mathbb{N}$ with $f(n) = 0_2$

Because f is surjective,

$\exists f^{-1}(0), f^{-1}(1), \dots, f^{-1}(n)$

Because f is ~~o.p.~~ injective, all these are distinct

Because f is o.p.,

$f^{-1}(0), f^{-1}(1), \dots, f^{-1}(n)$

all less than n .

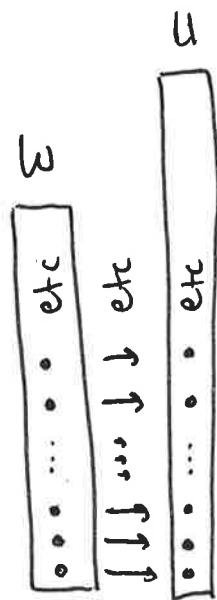
But this gives us $n+1$ natural numbers
less than $n!$

?Frustration, not factorial. $\Rightarrow \in \square$

Given wosets W and U , W is "smaller than" U if there exists an order-preserving map from W to U , but no order-preserving map from U to W .

How easy is this to check?

In practice, quite easy:



Match from the
bottom up.

Here U actually contains an order-isomorphic copy of W , right at the bottom.

Defn:

Let (ω, \prec) be a woset.

A subset $S \subseteq \omega$ is an initial segment of ω if whenever $x \in S$ and $y \prec x$, we have $y \in S$.

Eg: $\emptyset, \omega, \{y \in \omega : y \prec x\}$.

Theorem: Let $(U, \prec), (\omega, \triangleleft)$ be wosets. Then there exists an order-preserving map from U onto an initial segment of ω or an order-preserving map from ω onto an initial segment of V . Furthermore, this map is unique.