1 Introduction

My research interests lie at the intersection of algebra and combinatorics. I use tools from noncommutative algebra to study directed graphs. Given a particular kind of directed graph called a **layered graph**, we can construct a graded algebra $A(\Gamma)$, called the **splitting algebra**, or the **universal labeling algebra**. My research is focused on finding connections between the combinatorial structure of the graph and the structure of its labeling algebra.

This is a beautiful field of research, with connections to homological algebra, representation theory, and algebraic combinatorics. However, its grounding in finite structures makes it a rich source of collaborative projects with young mathematicians. There are open questions in this field that can be posed in the language of elementary linear algebra, or explored with basic combinatorial tools.

In the sections that follow, I will give a brief overview of this field of research, describe my recent work, and discuss my current research goals. I will include descriptions of my research projects with students from Canada/USA Mathcamp, and ideas for future projects I could explore with undergraduates.

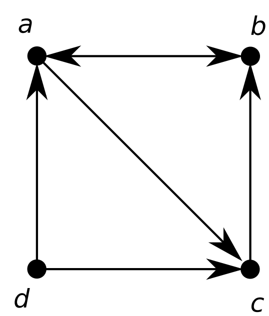
2 Elementary Background: Free Algebras and Directed Graphs

Noncommutative algebra studies algebraic structures which have an addition and multiplication operation, but in which the multiplication operation may not be commutative. Familiar examples include the quaternions and rings of matrices. Another example is $k\langle x_1, \ldots, x_n\rangle$, the **free algebra** on n generators. We can think of $k\langle x_1, \ldots, x_n\rangle$ as the set of polynomials in n non-commuting variables—that is, the monomials $x_i^2 x_j$, $x_i x_j x_i$, and $x_j x_i^2$ are all distinct. The free algebra is a nice example to start with because by imposing additional relations on the structure, we can create new algebras with their own interesting behaviors. For instance, if we start with $k\langle x, y, z\rangle$, the free algebra on three generators, we can impose the relation $xy = yx$, to obtain an algebra in which x and y commute with each other, but not with z. If we impose the relations

$$x^2 = y^2 = z^2 = xyz = -1,$$

then we obtain the quaternions.

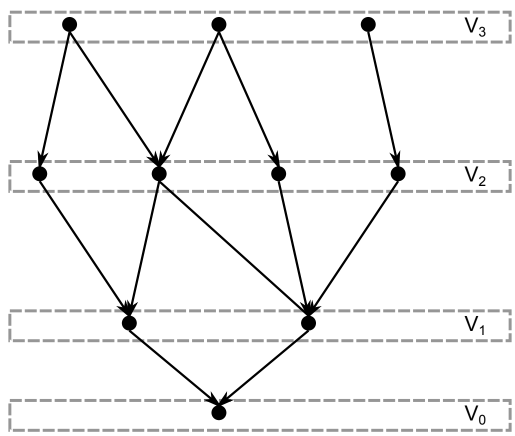
A **directed graph** consists of a set V of vertices, together with a set E of directed edges. These edges are ordered pairs of elements from V. So the ordered pair (v, w) is an edge from the vertex v to the vertex w. These edges are often represented graphically as arrows. For example, the graph given by $V = \{a, b, c, d\}$ and $E = \{(a, b), (a, c), (b, a), (c, b), (d, a), (d, c)\}$ corresponds to the picture below:



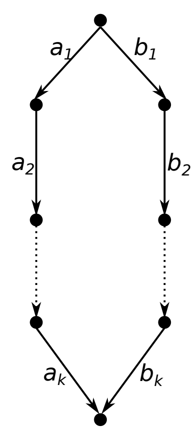
A **path** in a directed graph consists of an ordered list of edges of the form $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$. In our graphical interpretation, this corresponds to following a certain collection of arrows in order. For example, in the graph above, $(d, a), (a, c), (c, b), (b, a)$ would be the path that begins at the vertex d, then travels along the edge (d, a) to a, and then proceeds to c, then b, and finally comes to rest at a. The ordering of the edges in a path are important—if we rearrange the edges in the list above to get $(d, a), (c, b), (b, a), (a, c)$ this new list of edges no longer represents a path. In light of this, it makes sense to study directed graphs using noncommutative algebraic structures. If we consider the free algebra generated by the edges of a graph, the order of the multiplication gives us an algebraic interpretation of the idea of traveling through the edges of a directed graph in a particular order. We can then impose relations on this structure to reflect the behavior of the directed graph.

3 More Background: Layered Graphs And Associated Algebras

In [4], Gelfand, Retakh, Serconek, and Wilson define a **layered graph** to be a directed graph whose vertices can be partitioned into a finite collection of **layers** $V_0, V_1, V_2, \dots, V_n$ in such a way that every edge $(v, w) \in E$ travels one layer down. That is, if $v \in V_k$, then $w \in V_{k-1}$. An example of a layered graph is shown below:



To each layered graph $\Gamma = (V, E)$, the authors associate an algebra $A(\Gamma)$. This algebra is obtained by taking $k\langle E\rangle$, the free algebra generated by the edges of Γ, and imposing relations that reflect the idea that two paths through Γ should be considered "the same" if they have the same starting and ending vertices, as shown below:



Of course, there are many different ways we could interpret this notion of same-ness algebraically. If we consider the two paths $a_1, a_2, \dots, a_k$ and $b_1, b_2, \dots, b_k$, we could impose the relation

$$a_1 + a_2 + \ldots + a_k = b_1 + b_2 + \ldots + b_k,$$

or the relation

$$a_1 a_2 \ldots a_k = b_1 b_2 \ldots b_k.$$

Either of these takes algebraic representations of the paths in question and sets them equal to each other. However, what we will do is more general. We pass to the algebra $k\langle E\rangle[t]$, the free algebra generated by the edges, with a single central variable t adjoined. Then to the paths $a_1, a_2, \ldots a_k$ and $b_1, b_2, \ldots, b_k$ we associate the polynomials

$$(1 - ta_1)(1 - ta_2)\ldots(1 - ta_k)$$

and

$$(1 - tb_1)(1 - tb_2)\ldots(1 - tb_k),$$

respectively. Then in $k\langle E\rangle$, we impose all of the relations necessary to set the coefficients of these polynomials equal to each other. Notice that the degree-1 coefficient of this polynomial gives us the first relation, in which we take the sum of the edges, and the degree-k coefficient gives us the second relation, in which we take the product. There are also some more complicated relations associated to the other coefficients. We obtain an algebra $A(\Gamma) = k\langle E\rangle/R$, where R is the ideal generated by all relations of the form

$$\sum_{1\le i_1<i_2<\ldots<i_l\le k} a_{i_1}a_{i_2}\ldots a_{i_l} = \sum_{1\le i_1<i_2<\ldots<i_l\le k} b_{i_1}b_{i_2}\ldots b_{i_l},$$

for paths $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_k$ with the same starting and ending vertices, and $1 \le l \le k$.

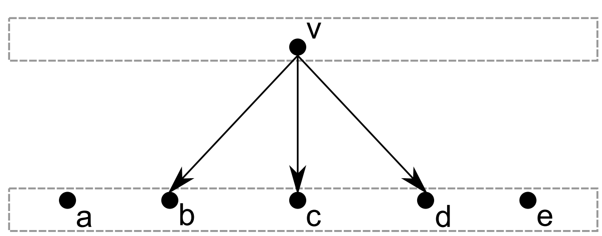
A second algebra called $B(\Gamma)$ often arises in the study of layered graphs, first introduced by Duffy in [1]. For a layered graph $\Gamma = (V, E)$, the algebra $B(\Gamma)$ is constructed by taking $k\langle V_+\rangle$, the free algebra generated by the vertices in $V \setminus V_0$, and quotienting out by the ideal R_B generated by

$$\{vw : (v, w) \notin E\} \bigcup \left\{ v \sum_{(v,w)\in E} w \right\}.$$

This algebra is easier to work with, since it has fewer generating relations, though the motivation behind its relations is not as clear. In Duffy's paper, she shows that in the case where Γ is a layered graph with a unique minimal vertex satisfying an extra condition called uniformity, the algebra $B(\Gamma)$ can be calculated directly from $A(\Gamma)$. However recently, papers such as [6] and [3] have dealt with $B(\Gamma)$ as an intrinsically interesting algebra in its own right.

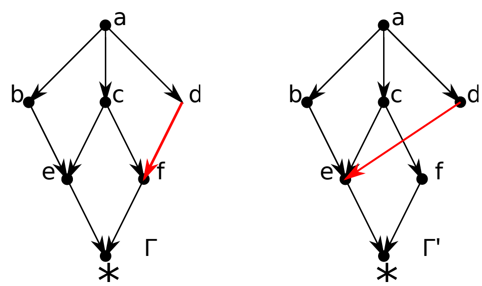
4 Recovering Graph Information From $B(\Gamma)$

One advantage to the algebra $B(\Gamma)$ is that its relations have a straightforward combinatorial interpretation. Given two vertices v and w, we have $vw = 0$ unless there is an edge from v to w. We also know that v multipled by the sum of all the vertices below it will be equal to zero. So in the example below, we will have $va = 0$, $ve = 0$, and $v(b + c + d) = 0$. We will also have $v^2 = 0$ and $vw = 0$ for any vertex w not included in the figure.



It is reasonable to ask: given just the graded algebra $B(\Gamma)$, how much information can we recover about the structure of the layered graph Γ? It turns out that in some cases, we can reconstruct the graph up to isomorphism. However, there are also cases in which nonisomorphic graphs will have the same graded algebra $B(\Gamma)$.

For example, consider the graphs Γ and Γ' pictured below. As directed graphs, they are clearly nonisomorphic—the graph Γ' has a vertex with in-degree three, while the graph Γ does not. However the collections of defining relations for the two graphs are the same: multiplying the vertex d on the left of any other vertex yields zero.



One major drive of my research is to discover which combinatorial properties of a graph Γ can be determined by looking at the algebra $B(\Gamma)$.

4.1 Recent Work

A part of my dissertation work was motivated by an attempt to show that given $B(\Gamma)$, we can find the collection of out-degrees of the vertices in any level of our graph. In [3], I define and construct an **upper vertex-like basis** for the algebra $B(\Gamma)$, which is a powerful tool for extracting combinatorial information from $B(\Gamma)$.

Formally, if $B(\Gamma)|_n$ is the subalgebra of $B(\Gamma)$ generated by $\bigcup_{i=1}^{n} V_i$, a basis L for $B(\Gamma)$ is upper vertex-like if for each n there exists an isomorphism of doubly-graded algebras

$$\phi : B(\Gamma)|_n \to B(\Gamma)|_n$$

such that ϕ fixes $\bigcup_{i=0}^{n-1} V_i$, and $\phi(V_n) = L$. An upper vertex-like basis for the n^{th} level of our graph is indistinguishable from the actual vertices if we work inside $B(\Gamma)|_n$. Our ability to construct such a basis allows us to pull quite a bit of useful information from $B(\Gamma)$. In particular, for each vertex $V \in V_+$, we can define a subspace

$$\kappa_v = \text{span}\left(\{w : (v,w) \notin E\} \cup \left\{\sum_{(v,w)\in E} w\right\}\right),$$

The upper-vertex-like basis allows us to find these subspaces for each level, and use them to calculate the out-degrees of the vertices. We can extract even more structural information by looking at the dimensions of the intersections of these spaces.

Upper vertex-like bases are particularly powerful for the class of non-nesting layered graphs. We say that Γ has the **non-nesting property** if for any two distinct vertices v and w with out-degree greater than one, we have

$$\{u : (v,u) \in E\} \not\subseteq \{u : (w,u) \in E\}$$

That is, the set of vertces with directed edges coming from v is never nested inside the set of vertices with directed edges coming from w. If Γ is a non-nesting layered graph with no vertices with out-degree one, we can use $B(\Gamma)$ to reconstruct Γ from the second layer up. That is, any graph Γ' satisfying $B(\Gamma') \cong B(\Gamma)$ will be isomorphic to Γ with the possible exception of the edges between V_2 and V_1.

This result, together with some additional combinatorial arguments allow me to show that certain classes of algebras can be uniquely identified by their algebras $B(\Gamma)$. These include the Hasse diagrams of the Boolean lattice, the lattice of subspaces of a finite-dimensional vector space over a finite field, and the complete layered graphs.

4.2 Work With Students

This past summer, I mentored a project with Jane Ahn, Ravi Movva, Nick White, and Sophia Xia at Canada/USA Mathcamp. Our goal was to extend my work with non-nesting posets, in particular exploring cases in which the graph Γ had vertices with out-degree one. In exploring the algebras, they discovered a way of counting the number of **branching paths**—paths in which all but the last vertex have out-degree two or more—between any two levels.

Using this in combination with upper vertex-like bases, we explored the Hasse diagram of the poset of factors of a natural number n. We discovered that in the case of a natural number of the form p^2q^3, where p and q are distinct primes, the graph is uniquely determined by the algebra $B(\Gamma)$. We are currently preparing these results for submission to an undergraduate publication.

4.3 Future Work

It would be interesting to explore this type of result fully for the simple case of a layered graph with only two nontrivial layers. In my dissertation, I did some work on this problem, and discovered some sufficient conditions for two-layer layered graphs to have isomorphic $B(\Gamma)$-algebras. I would like to find a set of conditions that are both necessary and sufficient. This is a problem that could be stated entirely in the language of linear algebra, and so it could potentially be tackled by an undergraduate.

5 Möbius Polynomials and Hilbert Series

Given a finite partially orderd set $(P, \leq)$ and elements $p, q \in P$, we define the **closed interval** $[p, q]$ to be the set of all elements $r \in P$ satisfying $p \leq r \leq q$. If $p \not\leq q$, then $[p, q]$ is empty. We use the notation I_P for the set of all closed intervals in P. We can define the **Möbius function** $\mu : I_P \to \mathbb{Z}$ recursively as follows:

$$\mu([p,q]) = \begin{cases} 1 & \text{if } p = q \\ -\sum_{p \leq r < q} \mu([p,r]) & \text{if } p < q \\ 0 & \text{if } p \not\leq q \end{cases}$$

Now we equip our poset P with a function $|\cdot| : P \to \mathbb{N}$, defined so that whenever $p < q$, we have $|p| < |q|$. For example, given a layered graph $\Gamma = (V, E)$, we can define a partial order relation $<$ on the vertices, by taking $v < w$ if and only if there exists a path from v to w in the graph Γ. In this way, we can view V as a partially ordered set with rank function given by $|v| = i$ if and only if $v \in V_i$. We define the **Möbius polynomial** of such a ranked poset to be

$$\mathcal{M}_P(z) = \sum_{p \leq q \in P} \mu([p,q]) z^{|q|-|p|}.$$

The Möbius polynomial is important in the calculation of the Hilbert series of the splitting algebra $A(\Gamma)$.

A Hilbert series is an algebraic invariant that can be calculated for any graded algebra with finite-dimensional graded pieces. In particular, if $A \cong k\langle x_1, \ldots, x_n\rangle / R$, where R is a homogeneous ideal in the graded free algebra $k\langle x_1, \ldots, x_n\rangle$, we can form a basis for the i^{th} graded piece by taking a maximal collection of degree-i monomials that are linearly independent of each other. This basis will be finite, since there are only i^n degree-i monomials in n variables. We can consider the generating function of the dimensions of these graded pieces:

$$H(A, z) = \sum_{i=0}^{\infty} \dim(A_i) z^i.$$

This is called the **Hilbert series** of the graded algebra A. Since the Hilbert series is a power series, it is possible for it to have a nice closed form. For instance, the Hilbert series for the free algebra $k\langle x_1, \ldots x_n\rangle$ is given by

$$H(k\langle x_1, \dots x_n\rangle, z) = \sum_{i=0}^{\infty} n^i z^i = \frac{1}{1-nz}.$$

This closed formula allows us to capture all the information about the dimension of the graded pieces of A in a compact way.

We may ask the question: given a graph Γ, what is the Hilbert series of $A(\Gamma)$? In the case where Γ is a layered graph with a unique minimal vertex, [7] gives the following formula for the Hilbert series:

$$H(A(\Gamma), z) = \frac{1-z}{1 - z\mathcal{M}_\Gamma(z)},$$

where $\mathcal{M}_\Gamma(z)$ is the Möbius polynomial defined above, applied to the poset of vertices of $Gamma$, with ordering generated by taking $v \geq w$ whenever $(v, w) \in E$.

5.1 Recent Work

In my dissertation, I calculated the Hilbert series and graded trace generating functions for the layered graphs corresponding to the lattices of Young diagrams which fit inside an $n \times n$ square.

In [2], I consider the Hilbert series associated to a direct product of ranked posets. The primary result in the paper is a simple fact about the Möbius polynomials: If P and Q are ranked posets, then

$$\mathcal{M}_{P\times Q}(z) = \mathcal{M}_P(z)\mathcal{M}_Q(z).$$

This allows me to calculate the Hilbert series and graded trace generating functions associated to the poset of positive factors of a natural number n, ordered by divisibility and ranked by number of prime factors, counted with multiplicity.

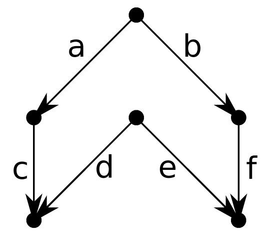
5.2 Work With Students

Working with Hilbert series and generating functions requires a fair amount of mathematical background. However, the Möbius polynomial is an easy invariant to calculate. Students who are new to mathematical research can make a lot of progress finding patterns and calculating Möbius polynomials for various important classes of examples. In 2012, Wai Shan Chan and Cary schneider at the University of Wisconsin, Eau Claire, calculated the Hilbert series of the face poset of the hypercube under the guidance of Colleen Duffy.

In a recent project with Meena Jagadeesen at Canada/USA Mathcamp we explored the Möbius polynomials of face posets of other convex polytopes. The results have been sumbitted for publication, and are available on arXiv in [5]. In this paper, we find formulas for the Möbius polynomials of face posets of prisms and pyramids over polytopes with known Möbius polynomials. We also present a formula for the Möbius polynomial of an arbitrary simplicial poset or an Eulerian poset using the number of faces of a given degree and some additional constraints.

5.3 Future Work

The formula from [7] for calculating the Hilbert series of $A(\Gamma)$ only works in the case where the layered graph has a unique minimal vertex. In the case where there are multiple minimal vertices, the formula breaks down. However, we could consider an alternate formulation of the relations that yield $A(\Gamma)$. Rather than considering pairs of paths that begin and end in the same place, we can consider closed loops in the graph, allowing ourselves to travel backwards along a particular edge.



For example, the loop in the graph above obtained by traveling forwards along edges a and c, then backwards along d, forwards along e, and thne backwards along f and b can be represented in the augmented algebra $k\langle E\rangle[[t]]$ by

$$(1-ta)(1-tc)(1-td)^{-1}(1-te)(1-tf)^{-1}(1-tb)^{-1}.$$

We require that all coefficients of this power series other than the constant term be equal to zero. This collection of relations yields the same algebra $A(\Gamma)$ in the case where Γ has a unique minimal vertex. Some experimentation with examples suggests that this new formulation preserves the relationship between the Hilbert series and the Möbius polynomial. Again, there is work that a bright undergraduate could do, working examples, and finding bases for graded pieces of this new algebra. However, the end goal of the project is to find a way of calculating a basis for this algebra and to determine whether this nice relationship between the Hilbert series and the Möbius polynomial exists in all cases.

6 Quadradicity of $A(\Gamma)$

A finitely-generated algebra is called **quadratic** if it can be expressed as the quotient of a free algebra by a homogeneous ideal generated by degree-two elements. One question we could consider is: under what circumstances is $A(\Gamma)$ quadratic. In [8], Shelton shows that the associated graded algebra $\mathrm{gr}A(\Gamma)$ is quadratic if and only if it satisfies a condition called **uniformity**. A uniform layered graph always has a quadratic $A(\Gamma)$, as well, but in this case, uniformity is not a necessary condition. I have found a weaker set of conditions that are also sufficient, and an even weaker set of conditions which I can prove to be necessary, but not sufficient. My goal in this area is to find necessary and sufficient conditions on the structure of the graph Γ to have a quadratic algebra $A(\Gamma)$.

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