# 1 The Vitali Set

## 1.1 Basic Cardinality Facts

This class lists basic facts about cardinality as a prerequisite. Just so everyone's on the same page, here's what you need to know:

Two sets $A$ and $B$ have the same **cardinality**, or are **equinumerous** if there exists a bijection between the two sets. Sometimes it's hard to find a bijection, but there's a result called the **Cantor-Bernstein** or **Schroeder-Bernstein** theorem[1], which says that if there exists an injection from $A$ to $B$, and another injection from $B$ to $A$, then a bijection exists.

A set is **countably infinite** if it is the same size as $\mathbb{N}$. Sets that are countably infinite include $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$.

A set is **the size of the continuum** if it is the same size as $\mathbb{R}$. Sets that are continuum-sized include $\mathbb{R}$, $[0, 1]$, $(0, 1)$, and $\mathscr{P}(\mathbb{N})$.

The powerset $\mathscr{P}(A)$ of a set $A$ always has a larger cardinaltiy than $A$. We have $|\mathscr{P}(\mathbb{N})| = |\mathbb{R}|$, which gives us $|\mathbb{N}| < |\mathbb{R}|$, but are there any cardinalities in between the two? This question was open for about a hundred years before it was proven to be independent of ZFC. In this class, we'll be exploring set theoretic universes in which intermediate cardinalities exist.

Good to go? Right—onward!

## 1.2 How Do We Measure Length?

If we draw a picture of the real line, it looks kind of like a big ruler extending infinitely in either direction, with a zero in the middle. If we extend this analogy, we have a nice intuitive way to measure the size of sets. The size of the interval $(a, b)$ is $b - a$. Suppose we tried to extend this notion of length to a function $\mathscr{M} : \mathscr{P}(\mathbb{R}) \to [0, \infty]$.[2] What sorts of properties would this function need to satisfy?

[1] It gets called both, but it's okay, because as far as I can tell the result is actually due entirely to Bernstein.

[2] How often have you seen a closed bracket at the $\infty$ end? Really what's going on is that the size of a given subset of $\mathbb{R}$ could be anything in $[0, \infty)$, or it could be infinite. So we get to close the bracket!

**Properties:**

- Intervals have the correct length. That is, $\mathscr{M}((a,b)) = b - a$.
- Unions of disjoint sets have the correct length. That is, if $A \cap B = \emptyset$, then $\mathscr{M}(A \cup B) = \mathscr{M}(A) + \mathscr{M}(B)$.
- Smaller sets have smaller measures. That is, if $A \subseteq B$, then $\mathscr{M}(A) \leq \mathscr{M}(B)$.
- $\mathscr{M}$ is translation invariant. That is, if $A + t = \{a + t \mid a \in A\}$, then $\mathscr{M}(A + t) = \mathscr{M}(A)$.

All of these are things that any reasonable notion of measure will satisfy. Unfortunately, as we're about to see, we've already caused ourselves a fairly serious problem.

## 1.3 The Vitali Set

We define a relation $\sim$ on the set of real numbers so that $r \sim s$ if and only if $(r - s) \in \mathbb{Q}$. It is easy to show that this is an equivalence relation:

**Reflexive:** Since $(r - r) = 0 \in \mathbb{Q}$ for any $r \in \mathbb{R}$, we have $r \sim r$ for all $r \in \mathbb{R}$.

**Symmetric:** If $r \sim s$, then $(r - s) \in \mathbb{Q}$, so $(s - r) \in \mathbb{Q}$, and so $s \sim r$.

**Transitive:** If $r \sim s$ and $s \sim t$, then $(r - s)$ and $(s - t)$ are both in $\mathbb{Q}$, so $(r - s) + (s - t) = r - t \in \mathbb{Q}$. So $r \sim t$.

Since $\sim$ is an equivalence relation, it breaks $\mathbb{R}$ into equivalence classes. Let $S_r$ be the equivalence class of the real number $r$. Clearly every $S_r$ will have nontrivial intersection with the open interval $(0, 1)$. Define a set $V$ by picking one element from each distinct $S_r \cap (0, 1)$, so that $|V \cap S_r| = 1$ for all $r \in \mathbb{R}$, and $V \subseteq (0, 1)$. Since $V \subseteq (0, 1)$, we know right off the bat that $0 \leq \mathscr{M}(V) \leq 1$. But can we narrow that down at all?

We define a set $U$, given by

$$U = \bigcup_{q \in (\mathbb{Q} \cap (-1,1))} (V + q)$$

That is, we take all of the rational numbers $q$ in the interval $(-1, 1)$, translate $V$ by those rational numbers, and take the union of the resulting sets.

This is a union of infinitely many disjoint translates of $V$, so if $\mathscr{M}(V)$ is nonzero, then $\mathscr{M}(U)$ will be infinite. Of course, we know that $V \subseteq (0,1)$, so it follows that $U \subseteq (-1,2)$. This means that $\mathscr{M}(U) \leq 3$. So we can conclude that $\mathscr{M}(V) = 0$.

## 1.4 Or Can We?

Except here's the problem. Since $\mathscr{M}(V) = 0$, and $U$ is a union of countably many translates of $V$, it must be the case that $\mathscr{M}(U) = 0$. But suppose $r$ is an arbitrary number in $(0,1)$. Then $r \in S_s$ for some $s \in V$, since $V$ has representatives from every equivalence class. Thus $s - r \in \mathbb{Q}$. Also, since $r \in (0,1)$ and $s \in (0,1)$, it must be the case that $|s-r| < 1$. So $(r-s)$ is a rational number in $(-1,1)$, and so $V + (r-s) \subseteq U$. In particular, $s + (r-s) = r \in U$. This shows that $(0,1) \subseteq U$. This tells us that $\mathscr{M}(U) \geq 1$.

## 1.5 So What Can We Conclude?

Basically, even our very simple assumptions about how a measure function ought to behave are fundamentally broken in some way. In particular, if you believe in the Axiom of Choice,[3] then whenever you have a measure function that satisfies our basic properties, it is possible to find a subset of $\mathbb{R}$ that cannot be measured using that property. there is no way to have a measure function that satisfies all of those properties. Our solution is to be a little more careful in how we define our function, so that it is clear that it is not defined on pathological sets like $V$.

If you want more detail about how, exactly, this works, you should definitely take Alfonso and Steve's class in Week 3. For this class, it will suffice to figure out how our measure works on **open sets**.

**Definition:** A subset $U \subseteq \mathbb{R}$ is **open** if for any point $u \in U$ there exists $\epsilon > 0$ such that the open interval $(u - \epsilon, u + \epsilon) \subseteq U$.

Open sets are easy to measure, because they can be decomposed into open intervals:

**Theorem:** A subset $U \subseteq \mathbb{R}$ is open if and only if it is the union of a countable number of open intervals.

[3] And I know you all do!

The proof of this theorem is left as an exercise. The following lemma will be helpful for one direction:

**Lemma:** Any union of open sets is an open set.

*Proof.* Let $\{U_\alpha : \alpha \in A\}$ be a set of open sets. Let $U = \bigcup_{\alpha \in A} U_\alpha$. We wish to show that $U$ is open. Suppose $u \in U$. Then $u \in U_\alpha$ for some $\alpha \in A$, and so there exists $\epsilon > 0$ such that $(u - \epsilon, u + \epsilon) \subseteq U_\alpha \subseteq U$. □

For this class, we'll assume that we have a measure function

$$\mu : \text{Measurable Sets} \to [0, \infty]$$

that satisfies the following properties:

- The function $\mu$ is defined for all open intervals, and $\mu((a, b)) = b - a$.
- Let $U$ be a countable union of nonempty open intervals. That is

  $$U = \bigcup_{n=1}^{\infty} I_n.$$

  We also call such a set an **open** set. Then $\mu(U) = \sum_{n=1}^{\infty} \mu(I_n)$.
- A set $S$ satisfies $\mu(S) = 0$ if and only if for any $\epsilon > 0$, there exists an open set $U \supseteq S$ such that $\mu(U) \leq \epsilon$. In this case we call $S$ a **measure-zero set**, or a **null set**.

In this class our primary interest is in measure-zero sets.

**Example 1:** Any one-point set $\{r\}$ has measure zero.

*Proof.* Fix $\epsilon > 0$. Then let $U$ be the open interval of length $\epsilon$ centered at $r$. □

**Example 2:** Any countable set has measure zero.

*Proof.* Let $S = \{s_n : n \in \mathbb{N}\}$. Fix $\epsilon > 0$. For each $n$, let $I_n$ be the interval of length $\frac{\epsilon}{2^n}$, centered at $s_n$, and let $U \bigcup_{n=1}^{\infty} I_n$. Every element of $S$ is covered by $U$, and

$$\mu(U) = \sum_{n=1}^{\infty} \mu(I_n) = \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon.$$

By definition, this means that $S$ has measure zero. □

**Theorem:** A countable union of measure-zero sets has measure zero.

*Proof.* Let $S = \bigcup_{n=1}^{\infty} S_n$, such that each $S_n$ is a measure-zero set. Fix $\epsilon > 0$.

For each $n$, we know that $S_n$ is measure zero. Thus there exists $U_n \supseteq S_n$, with $\mu(U_n) \leq \frac{\epsilon}{2^n}$. We define $U = \bigcup_{n=1}^{\infty} U_n$.

Since $U_n$ is open, we have $U_n = \bigcup_{k=1}^{\infty} I_{n,k}$. This means that

$$U = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} I_{n,k}.$$

If all the $I_{n,k}$ are disjoint, then $\mu(U)$ will be the sum of their lengths. Otherwise, the intervals overlap, and $U$ can be covered with a set of intervals with a smaller total length. In either case, we have:

$$\mu(U) \leq \sum_{n,k \in \mathbb{N}} \mu(I_{n,k}) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \mu(I_{n,k}) = \sum_{n=1}^{\infty} \mu(U_n) \leq \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon.$$

Thus by definition, $S$ has measure zero. $\square$

Of course, if we take continuum many measure zero sets, this will no longer work. We could simply take the union of all the one-point sets that make up $(0, 1)$, which clearly cannot be covered with an interval of length $1/2$. Or worse, we could take the union of all the one-point sets everywhere to get the real line.

But here's an interesting question: What if we took uncounably many measure zero sets, but not continuum many? What happens if we take one of the infinities in between? This turns out to be independent of ZFC+$\neg$CH. Over the course of the week we'll introduce an extra-set-theoretic axiom called Martin's Axiom which will allow us to explore this question in a meaningful way.

## 1.6 Homework!

1. (a) Prove that $|\mathscr{P}(\mathbb{N})| = |\mathbb{R}|$.

   (b) Consider $\{f : \mathbb{N} \to \mathbb{N}\}$, the set of functions from $\mathbb{N}$ to $\mathbb{N}$. Prove that $|\{f : \mathbb{N} \to \mathbb{N}\}| = |\mathbb{R}|$.

2. Prove the theorem stated in class: that $U$ is open if and only if it is the disjoint union of countably many open intervals.

3. Consider the Cantor middle thirds set. This is the set constructed by starting with the interval $[0, 1]$. We remove the middle third to get $[0, 1/3] \cup [2/3, 1]$. Then we remove the middle two third of the two remaining segments to get

$$[0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$$

   We repeat this process countably many times.

   (a) Show that the resulting set has measure zero.

   (b) Show that this gives us an example of an uncountable measure-zero set.

# 2 The Dominating Function Question

## 2.1 Dominating Functions

Let $f$ and $g$ be functions from $\mathbb{N}$ to $\mathbb{N}$. We say $g$ **dominates** $f$ and write $g >_* f$ if there exists $n \in \mathbb{N}$ such that for any $m > n$, we have $g(m) > f(m)$. That is, the function $g$ is *eventually bigger* than the function $f$.

Suppose we have a set of functions $\mathscr{F} = \{f_n \mid n \in \mathbb{N}\}$ from $\mathbb{N}$ to $\mathbb{N}$. We ask the following questions:

**Warmup Question:** Does there necessarily exist a function $g$ such that $f_n <_* g$ for all $n \in \mathbb{N}$?

**Answer:** Yes! Take $g(n) = \max\limits_{i<n}\{f_i(n)\} + 1$.

**Slightly Harder Question:** What if we take $\mathscr{F} = \{f_r : r \in \mathbb{R}\}$?

**Answer:** Not necessarily. $\mathbb{R}$ is equinumerous with the set of functions from $\mathbb{N}$ to $\mathbb{N}$, so $\mathscr{F}$ could be the set of all functions. That'd be pretty hard to dominate.

**Our Question For Today:** What if $|\mathbb{N}| < |\mathscr{F}| < |\mathbb{R}|$?

This is another question independent of ZFC+$\neg$CH which can be attacked by Martin's axiom. And it's a pretty good warm up for the measure-zero question we asked yesterday. Our goal for the next two days is to show that, assuming Martin's Axiom, we can construct a function $g$ which dominates every $f \in \mathscr{F}$.

## 2.2 Recycling the Previous Proof

The proof that we used in the countable case clearly doesn't work here, but there are pieces we can recycle if we think carefully about what, exactly is going on in the proof. This is a constructive proof by recursion on the natural numbers. All constructions like this follow a particular pattern:

- We want to construct an object $X$ that behaves in a particular way with respect to every natural number $n$.
- In the $n$'th step, we guarantee that $X$ behaves nicely with respect to $n$.

Our construction was fairly compact, so it's a little difficult to see exactly how these steps fit. Let's try to break it down:

- We want to construct a function $g$ which dominates every function $f_n$.
- On the $n$'th step, we do two things:
  - We choose a value for $g(n)$.
  - We *promise* that from now on, whenever we pick a value for $g(k)$, it will be larger than $f_n(k)$.

Choosing a value here was the obvious step—we have to choose values to construct a function, after all. But the important step—the thing that really makes this construction work—is the promises that we're making as we go! That's the step where we guarantee that $g$ is going to dominate $f_n$. For each $f_n$, I know that if I take any $m > n$, I will have $g(m) > f_n(m)$, because in the $n$'th step I promised that I would make that happen.

This sort of recursive construction is great for performing a countable number of tasks in a countable number of steps. Here the "tasks" are making sure that $g$ dominates each one of the functions $f_n$. The countability is nice, because it allows us to proceed linearly. At each step, we add one additional piece of information about our function $g$ (Namely, the value of $g(n)$), and when we're all done, we've constructed a complete function $g : \mathbb{N} \to \mathbb{N}$ that does exactly what we want it to do.

Unfortunately, when we lose countability, we lose the ability to proceed through each step in the construction in order. So if we can't use a linear order, what's the next best thing?

## 2.3 A Partial Order!

So! We want to build a function that dominates uncountably many functions $f_\alpha$, but we can't proceed linearly. We know that over the course of the construction, we will be doing two things: choosing values, and making promises. This means that if I press pause midway through the process, I'm going to have:

- A finite piece of a function $\phi : \{1, 2, \dots, n\} \to \mathbb{N}$.
- A finite set of promises we've made, represented by a subset $\mathscr{F}_0$ of $\mathscr{F}$, the set of functions we intend to dominate.

Whenever we extend our partial function $\phi$ to get a bigger piece of $g$, we keep our promises by making sure that the new values $g(n)$ are larger than the corresponding $f_\alpha(n)$ for all $f_\alpha \in \mathscr{F}_0$.

For example, suppose we have an ordered pair $(\phi, \mathscr{F}_0)$, with

$$\begin{aligned} \phi: \quad 0 &\mapsto 5 \\ 1 &\mapsto 3 \\ 2 &\mapsto 4 \end{aligned}$$

and with $\mathscr{F}_0 = \{f, g, h\}$, such that $f(n) = n$, $g(n) = n^n$, and $h(n) = 100$. We think of this ordered pair as a freeze-frame in the construction of our function $g$. At this moment, we've chosen values at 0, 1, and 2. When we choose a value for 3, it needs to be greater that $f(3) = 3$, $g(3) = 27$, and $h(3) = 100$. So we can choose any number greater than 100.

We can arrange these freeze-frames into a poset $\mathbb{P}_{\mathscr{F}}$, with ordering relation given by $(\phi, \mathscr{F}_0) \geq (\psi, \mathscr{F}_1)$ if and only if

(i) $\psi$ is an extension of $\phi$.

(ii) $\mathscr{F}_0 \subseteq \mathscr{F}_1$.

(iii) For any $f \in \mathscr{F}_0$ and for any $m \in (\mathrm{dom}(\psi) \setminus \mathrm{dom}(\phi))$, we have $\psi(m) > f(m)$.

Roughly speaking, $(\phi, \mathscr{F}_0) \geq (\psi, \mathscr{F}_1)$ if $(\psi, \mathscr{F}_1)$ is a valid extension of $(\phi, \mathscr{F}_0)$.[4] The first condition ensures that in moving from $(\phi, \mathscr{F}_0)$ to $(\psi, \mathscr{F}_1)$, we haven't changed any of the values that we had already defined at stage $(\phi, \mathscr{F}_0)$. The second condition ensures that we haven't removed any of the promises we'd made at stage $(\phi, \mathscr{F}_0)$.[5] The third condition ensures that as we extend $\phi$ to obtain $\psi$, we honor all of the promes that were made in $(\phi, \mathscr{F}_0)$. We say that $(\psi, \mathscr{F}_1)$ is a **strengthening** of $(\phi, \mathscr{F}_0)$.

If $(\phi, \mathscr{F}_0) \geq (\psi, \mathscr{F}_1)$ and $(\varphi, \mathscr{F}_2) \geq (\psi, \mathscr{F}_1)$, then we call $(\psi, \mathscr{F}_1)$ a **common strengthening** of $(\phi, \mathscr{F}_0)$ and $(\varphi, \mathscr{F}_2)$. In this case, we say that $(\phi, \mathscr{F}_0)$ and $(\varphi, \mathscr{F}_2)$ are **compatible**, since they could both be freeze-frames in the construction of the same function. Two elements $p$ and $q$ that are not compatible are called **incompatible**, and we write $p \perp q$.

[4]This may seem somewhat backwards, since the "larger" ordered pair actually gives us a smaller piece of the function. Try thinking about it like this: the ordered pair represents the collection of all possible functions that could result from this freeze-frame appearing in the construction. Extending the function or making more promises makes the number of possible functions smaller.

[5]No takebacks!

## 2.4 We Found A Dominating Function!

Just kidding, we totally didn't. But let's pretend that we did. Assume that there exists a function $g$ such that $g >_* f$ for every $f \in \mathscr{F}$. Consider the set $G \subseteq \mathbb{P}_{\mathscr{F}}$ such that for every $(\phi, \mathscr{F}_0) \in G$,

(i) $g$ is an extension of $\phi$.

(ii) For any $f \in \mathscr{F}$ and for any $m \notin \mathrm{dom}(\phi)$, we have $g(m) > f(m)$.

That is to say, $G$ consists of all ordered pairs representing freeze-frames that could appear in the construction of the function $g$. Notice that $G$ has several nice properties:

(a) If $p$ and $q$ are both in $G$, then there exists $r \in G$ with $r \leq p$ and $r \leq q$. That is, all elements of $G$ are compatible.

(b) For all $p$ and $q$ in $\mathbb{P}_{\mathscr{F}}$ such that $q \geq p \in G$, we have $q$ in $G$. We say that the set $G$ is "closed upwards" in $\mathbb{P}_{\mathscr{F}}$.

Any subset $X$ of a poset $\mathbb{P}$ that satisfies conditions $(a)$ and $(b)$ is called a **filter**. We also have two additional properties:

(c) For each $n \in \mathbb{N}$, there exists $(\phi, \mathscr{F}_0) \in G$ such that $n \in \mathrm{dom}(\phi)$.

(d) For each $f \in \mathscr{F}$, there exists $(\phi, \mathscr{F}_0) \in G$ such that $f \in \mathscr{F}_0$.

Tomorrow we'll see that any subset of $\mathbb{P}_{\mathscr{F}}$ that satisfies these four properties can actually be used to build a function $g$ that dominates every function if $\mathscr{F}$.

## 2.5 Homework!

1. Suppose $\mathscr{F}$ is the set of constant functions and $g(n) = 2n$.
   (a) Show that $g >_* f$ for all $f \in \mathscr{F}$.
   (b) What does the filter $G$ from section LABEL look like? find necessary and sufficient conditions for an element $(\phi, \mathscr{F}_0)$ of the poset $\mathbb{P}_{\mathscr{F}}$ to ge in $G$.

2. Given below are four different elements of $\mathbb{P}_{\mathscr{F}}$. Decide which pairs are compatible, and which are incompatible. If possible, find a common strengthening.

$$\phi_1 : \begin{cases} 0 \to 1 \\ 1 \to 4 \\ 2 \to 6 \end{cases}, \qquad \begin{array}{c} \mathscr{F}_1 = \{a, b\} \\ a(n) = 2 \\ b(n) = n^2 \end{array}$$

$$\phi_2 : \begin{cases} 0 \to 1 \end{cases}, \qquad \begin{array}{c} \mathscr{F}_2 = \{c\} \\ c(n) = 3n \end{array}$$

$$\phi_3 : \begin{cases} 0 \to 0 \\ 1 \to 4 \end{cases}, \qquad \begin{array}{c} \mathscr{F}_3 = \{a, d\} \\ a(n) = 2 \\ d(n) = n + 2 \end{array}$$

$$\phi_4 : \begin{cases} 0 \to 47 \end{cases}, \qquad \mathscr{F}_4 = \emptyset$$

3. Let $\mathbb{P}$ be a poset, and let $p \in \mathbb{P}$. Is there necessarily a filter in $\mathbb{P}$ containing $p$? Either describe such a filter, or explain why it cannot exist.

4. A subset $D$ of a poset $\mathbb{P}$ is called **dense** if for any $p \in \mathbb{P}$, there exists $q \leq p$ such that $q \in D$. Show that the following sets are dense in $\mathbb{P}_{\mathscr{F}}$:
   (a) The set of $(\phi, \mathscr{F}_0)$ such that $1,000 \in \mathrm{dom}(\phi)$.
   (b) The set of $(\phi, \mathscr{F}_0)$ such that for a particular $f \in \mathscr{F}$, we have $f \in \mathscr{F}_0$.

# 3 Day 3 Notes

## 3.1 Filters Build Functions

Okay, that was fun. Now let's take a step back. Instead of assuming our goal-function $g$ exists, let's instead assume that there exists a subset $G$ of $\mathbb{P}_{\mathscr{F}}$ satisfying properties (a), (b), (c), and (d). that is, there exists a filter $G$ that intersects with every $D_n$ and with every $D_f$.

**Claim:** $G$ allows us to build a function $g$ which will dominate every $f \in \mathscr{F}$.

*Proof.* We define $g$ to be equal to

$$\bigcup \{\phi \mid \exists \mathscr{F}_0 \subseteq \mathscr{F} \text{ such that } (\phi, \mathscr{F}_0) \in G\}$$ [6]

We need to show that this gives us a well defined function $g$ on all of $\mathbb{N}$ that dominates $f$ for every $f \in \mathscr{F}$.

Let's start by showing that $g$ is well-defined. Suppose $(\phi, \mathscr{F}_0)$ and $(\psi, \mathscr{F}_1)$ are both in $G$, and that both $\phi$ and $\psi$ are defined on $n$. We want to show that $\phi(n) = \psi(n)$. We know that $(\phi, \mathscr{F}_0)$ and $(\psi, \mathscr{F}_1)$ have a common strengthening: call it $(\Phi, \mathscr{F}_2)$. Then $\phi(n) = \Phi(n)$ and $\psi(n) = \Phi(n)$, so we must have $\phi(n) = \psi(n)$.

Now we'll show that $g$ is defined on all of $\mathbb{N}$. For any $n \in \mathbb{N}$, we have $G \cap D_n \neq \emptyset$. So there exists $(\phi_n, \mathscr{F}_n) \in G$ such that $\phi_n$ is defined at $n$. This means that $g$ is defined at $n$.

Finally, we'll show that $g >_* f$ for all $f \in \mathscr{F}$. We know that for any $f \in \mathscr{F}$, we have $G \cap D_f \neq \emptyset$. So there exists $(\phi_f, \mathscr{F}_f) \in G$ such that $f \in \mathscr{F}_f$. Let $\text{dom}(\phi_f) = \{1, 2, \ldots, n\}$. We claim that for any $m > n$, we have $g(m) > f(m)$. If we can prove this claim, we will have shown that $g > *f$.

The value of $g(m)$ will be given by any element $(\phi_m, \mathscr{F}_m) \in D_m$. We know that $(\phi_m, \mathscr{F}_m)$ and $(\phi_f, \mathscr{F}_f)$ are compatible, so they have a common strengthening—let's call it $(\phi_{mf}, \mathscr{F}_{mf})$. Since $(\phi_{mf}, \mathscr{F}_{mf})$ is a strengthening of $(\phi_m, \mathscr{F}_m)$, it is defined at $m$, and $g(m) = \phi_{mf}(m)$. Since $(\phi_{mf}, \mathscr{F}_{mf})$ is a strengthening of $(\phi_f, \mathscr{F}_f)$, we know that $\phi_{mf}(m) > f(m)$. So we're done! Yaaay! □

[6] Oooohkay. So this is technically correct if you think of a function as a set of ordered pairs. If you don't want to think of a function as a set of ordered pairs, then this means more or less what you'd expect—you glue the functions together. But this is a set theory class. Of *course* you want to think of a function as a set of ordered pairs.

## 3.2 Okay, But How Does This Help?

So we've shown that if there is a filter that intersects with these sets, then we can find a dominating function. But why should we believe that such a filter exists? Well, it seems not entirely implausable that such a thing *could* be true. The sets we have chosen for this intersection are in a technical sense quite large.

**Definition:** We say that a subset $D \subseteq \mathbb{P}$ is **dense** iff for any $p \in \mathbb{P}$, there exists $q \in D$ such that $q \leq p$.

So if we pick any element in $\mathbb{P}$, there will always be an element of the dense set below it.

**Stupid Example:** For any poset $\mathbb{P}$, $\mathbb{P}$ is dense in $\mathbb{P}$.[7]

**Less Stupid Example:** The set $D_f = \{\phi, \mathscr{F}_0) \mid f \in \mathrm{dom}(\phi)\}$ is dense in $\mathbb{P}_{\mathscr{F}}$.

**Another Relevant Example:** The set $D_n = \{(\phi, \mathscr{F}_0) \mid n \in \mathrm{dom}(\phi)\}$ is also dense in $\mathbb{P}_{\mathscr{F}}$.

Notice that property (c) says that for any $n \in \mathbb{N}$, $G \cap D_n$ is nonempty. Property (d) says that for any $f \in \mathscr{F}$, $G \cap D_f$ is nonempty. So essentially all that we need to do is find a filter that intersects with a large collection of very large sets. The only question is: how many of these very large sets can we reasonably expect a filter to intersect with?

**Fact 1:** (ZFC) If $\mathbb{P}$ is a poset and $\{D_n : n \in \mathbb{N}\}$ is a countable collection of dense sets, then there exists a filter $G \subseteq \mathbb{P}$ such that $G \cap D_n \neq \emptyset$ for all $n \in \mathbb{N}$.

*Proof.* Let $\mathbb{P}$ be a poset, and let $\{D_n : n \in \mathbb{N}\}$ be a countable collection of dense sets. Then there exists a sequence of elements

$$p_0 \leq p_1 \leq \ldots \leq p_n \leq \ldots$$

such that each $p_i$ is an element of $D_i$. The set $\{p_n \mid n \in \mathbb{N}\}$ is a filter. $\square$

On the other hand, if we insisted that $G$ intersect with continuum many dense sets, then we could use the same argument from section 3.1 to show that there exists a function that dominates every function from $\mathbb{N}$ to $\mathbb{N}$, which we know cannot exist. So again, we have a result which behaves very differently in the countable and continuous cases. Let's propose the following axiom:

[7] Yep. Told you it was going to be stupid.

**Proposed Axiom:** If $\mathbb{P}$ is a poset, $A$ is a set such that $|A| < |\mathbb{R}|$, and $\{D_\alpha : \alpha \in A\}$ is a collection of dense sets, then there exists a filter $G \subseteq \mathbb{P}$ such that $G \cap D_\alpha \neq \emptyset$.

### 3.3 This Axiom Is Dumb

What's wrong with this axiom? In short: it's not true. If we leave the axiom as written, then our filters are too powerful—they allow us to construct functions which simply cannot exist. Observe:

Define a poset $\mathbb{P}_{☹}$ consisting of partial functions $\phi : \{1, 2, \ldots, n\} \to \{\omega_1\}$, with ordering relation given by $\psi \leq \phi$ if $\psi$ is an extension of $\phi$. For each $\alpha < \omega_1$, define $D_\alpha = \{\phi \mid \alpha \in \mathrm{Im}(\phi)\}$. These sets are all dense. The dumb axiom tells us that there exists a filter $G$ that intersects with all of these dense sets. Define $g$ to be the union of all the functions in $G$. Let $\alpha < \omega_1$. Since $G \cap D_\alpha$ is nonempty, there exists $\phi \in G$ so that $\alpha \in \mathrm{Im}(\phi)$. Thus for some $n \in \mathbb{N}$, we have $g(n) = \alpha$. Therefore, $g(n)$ is a surjective function from $\mathbb{N}$ to $\omega_1$. This is what we in the world of mathematics refer to as a "porbelm".[8]

### 3.4 The Countable Chain Condition

Recall that $p$ and $q$ are **compatible** if there exists a common strengthening of $p$ and $q$. Otherwise, $p$ and $q$ are **incompatible**, and we write $p \perp q$.

We call a subset $A$ of a poset $\mathbb{P}$ an **antichain** if the elements of $A$ are pairwise incompatible. That is, for any $a, b \in A$, there is no $p \in \mathbb{P}$ satisfying $p \geq a$ and $p \geq b$.

If we think of a poset as a collection of puzzle pieces, then an antichain is a collection of pieces that all belong to different puzzles. There is no way of making a coherent whole out of them. Even if you have less than $2^{\aleph_0}$ dense sets, when the antichains become too large, it becomes impossible to find a filter that intersects with all of them. The common strengthening condition becomes too hard to satisfy with all of these incompatible choices floating around. That was the problem with our poset $\mathbb{P}_{☹}$. It has enormous (i.e. uncountable) antichains.

[8]**Definition**: A problem so terrible, it has broken the word "problem." Definitively *not* a word used by most mathematicians, whatever Susan might tell you.

We say that $\mathbb{P}$ satisfies the **countable chain condition** if every antichain of $\mathbb{P}$ is countable.[9] We also say "$\mathbb{P}$ is CCC."[10]

**Martin's Axiom:** (The Real Version) If $\mathbb{P}$ is a CCC poset, $A$ is a set such that $|A| < |\mathbb{R}|$, and $\{D_\alpha : \alpha \in A\}$ is a collection of dense sets, then there exists a filter $G \subseteq \mathbb{P}$ such that $G \cap D_\alpha \neq \emptyset$.

[9] Yes, this is clearly backwards and wrong. It should be the countable *antichain* condition. Don't blame me—I didn't name it.

[10] And yes, this reads "$\mathbb{P}$ is Countable Chain Condition." Really we should never allow logicians to name anything ever. **Optional Homework:** Tell Steve that I said so.

### 3.5 Homework!

1. Let $x$ be a nonempty set, and let $\mathbb{P}$ be the collection of nonempty subsets of $x$, with ordering given by set inclusion.

   (a) For $y, z \subset x$, when do we have $y \perp z$?

   (b) Can you find necessary and sufficient conditions for $\mathbb{P}$ to be CCC?

2. Let $|\mathbb{N}| < A < |\mathbb{R}|$. Prove that the poset of all functions

$$\phi : \{0, 1, 2, \ldots, n\} \to A$$

   is not CCC.

# 4 Day 4 Handount (Extra n's are always good!)

## 4.1 CCC Posets, Continued

Last time we introduced Martin's Axiom, which says that if $\mathbb{P}$ is a CCC poset, $A$ is a set such that $|A| < |\mathbb{R}|$, and $\{D_\alpha : \alpha \in A\}$ is a collection of dense sets, then there exists a filter $G \subseteq \mathbb{P}$ such that $G \cap D_\alpha \neq \emptyset$. We should probably verify that the poset $\mathbb{P}_{\mathscr{F}}$ from section 2.3 is CCC, and that the poset $\mathbb{P}_{☹}$ from section 2.3 is not. We'll start with $\mathbb{P}_{☹}$

**Proposition:** $\mathbb{P}_{☹}$ is not CCC.

*Proof.* The set $\{f_\alpha : 0 \mapsto \alpha : \alpha \in A\}$ is a n uncountable antichain. □

**Proposition:** $\mathbb{P}_{\mathscr{F}}$ is CCC.

*Proof.* There are only countably many finite partial functions $\phi$. By the pigeonhole principle, in any uncountable subset of $\mathbb{P}_{\mathscr{F}}$, there exist two elements $(\phi, \mathscr{F}_0)$ and $(\psi, \mathscr{F}_1)$ such that $\phi = \psi$. These two elements have a common strengthening given by $(\phi, \mathscr{F}_0 \cup \mathscr{F}_1)$. Thus no uncountable subset of $\mathbb{P}_{\mathscr{F}}$ is an antichain. It follows that all antichains of $\mathbb{P}_{\mathscr{F}}$ are countable. □

This finishes our discussion of dominating functions. Now that we've shown that $\mathbb{P}_{\mathscr{F}}$ is CCC, our argument from section 3.1 shows that we have the following:

**Theorem:** (MA) If $A$ is a set such that $|A| < |\mathbb{R}|$, and $\mathscr{F} = \{f_\alpha : \alpha \in A\}$ is a set of functions from $\mathbb{N}$ to $\mathbb{N}$, then there exists a function $g : \mathbb{N} \mapsto \mathbb{N}$ such that $g >_* f_\alpha$ for all $\alpha \in A$.

## 4.2 Back to Measure Theory!

When we first started talking about Dominating functions, we discussed this framework for a recursive construction that accomplishes a countable number of tasks in a countable number of steps:

**Recursive Construction:**

- We want to construct an object $X$ that behaves in a particular way with respect to every natural number $n$.

- In the $n$'th step, we guarantee that $X$ behaves nicely with respect to $n$.

Over the past couple of days, we've shown that we can use Martin's axiom to tweak this kind of construction so that we can perform an uncountable (but not continuum) number of tasks instead. Our new framework looks like this:

**Martin's Axiom Construction:**

- We want to construct an object $X$ that behaves in a particular way with respect to every $\alpha \in A$, for a set $A$ with $|A| < |\mathbb{R}|$.
- We use Martin's Axiom to find a filter that intersects with a dense set $D_\alpha$ for every $\alpha \in A$. Its intersection with $D_\alpha$ will guarantee that the object built by the filter behaves nicely with respect to $\alpha$.

Here's how our recursive construction worked for our proof that the union of countably many measure-zero sets is a measure-zero set.

**Recursive Construction:**

- We want to construct an open set $U$ with $\mu(U) \leq \epsilon$ that covers $S_n$ for every $n \in \mathbb{N}$.
- We start with $U_0 = \emptyset$.
- In the $n$'th step, we extend $U_{n-1}$ to a bigger open set $U_n$ with $\mu(U_n) < \epsilon$ and $S_n \subseteq U_n$.
- When we're done, we take the union of all the $U_n$.

Obviously, there are some details we left out here. For instance, we had to be pretty careful to make sure that the final set that we got—the union of all the $U_n$'s—was actually the right size. But we can sort of see how we might be able to transfer this framework over into a Martin's Axiom argument.

**Martin's Axiom Construction:**

- We want to construct an open set $U$ with $\mu(U) \leq \epsilon$ that covers $S_\alpha$ for every $\alpha \in A$, for a set $A$ with $|\mathbb{N}| < |A| < |\mathbb{R}|$.
- We use Martin's Axiom to find a filter that intersects with a dense set $D_\alpha$ for each $\alpha \in A$. Its intersection with $D_\alpha$ will guarantee that the set $U$ built by the filter will cover $S_\alpha$.

## 4.3 Constructing The Set $U$

Let $\mathbb{P} = \{W \subseteq \mathbb{R} \mid W \text{ is open and } \mu(W) < \epsilon\}$, ordered by $W \leq V$ iff $W \supseteq V$. For each $\alpha \in A$, let

$$D_\alpha = \{W : S_\alpha \subseteq W\}.$$

Suppose $V \in \mathbb{P}$, and $S_\alpha \not\subseteq V$. Since $S_\alpha$ is measure-zero, we can find an open set $X$ such that $\mu(X) < \epsilon - \mu(V)$, and $S_\alpha \subseteq X$. Then $W = V \cup X \in \mathbb{P}$, since $\mu(W) \leq \mu(X) + \mu(V) < \epsilon$. We have $S\alpha \subseteq W$, so $W \in D_\alpha$, and $V \subseteq W$, so $W < V$.

If we can find a filter in $\mathbb{P}$ that intersects with all of these $D_\alpha$'s, then the union of all of the sets in this filter will be an open set which covers every $S_\alpha$. And it seems like a good bet that this open set will have measure $\leq \epsilon$, though of course we'll have to verify that.

But before we get ahead of ourselves, let's remember: Martin's axiom will only find a filter for us if our poset is CCC.

**Theorem:** $\mathbb{P}$ is CCC.

*Proof.* Two sets $U$ and $V$ in $\mathbb{P}$ are incompatible if and only if $\mu(U \cup V) \geq \epsilon$. Let's assume that some uncountable antichain $\mathscr{A}$ exists. Let's define

$$\mathscr{A}_n = \left\{U \in \mathscr{A} : \mu(U) \leq \epsilon - \frac{3}{n}\right\}.$$

Then we have

$$\mathscr{A} = \bigcup_{n=1}^{\infty} \mathscr{A}_n.$$

The countable union of countable sets is countable, and $\mathscr{A}$ is uncountable. Thus it must be the case that there exists some $n \in \mathbb{N}$ such that $\mathscr{A}_n$ is an uncountable antichain.

Let $\mathscr{Q}$ be the set of open subsets of $\mathbb{R}$ obtained by taking finite unions of intervals with rational endpoints. This collection of sets is countable, and any open set $U$ with finite measure can be approximated arbitrarily closely by some open set in $\mathscr{Q}$. Or more precisely:

**Exercise:** If $U$ is an open set and $\mu(U) < \infty$, then for any $\epsilon > 0$, there exists $W \in \mathscr{Q}$ such that $\mu(U \Delta W) < \epsilon$.

We now define

$$\mathscr{A}_{n,W} = \left\{U \in \mathscr{A}_n : \mu(U \Delta W) \leq \frac{1}{n}\right\}.$$

The result stated above gives us

$$\mathscr{A}_n = \bigcup_{W \in \mathscr{Q}} A_{n,W},$$

and since $\mathscr{A}_n$ is uncountable and ther eare only countably many $W \in \mathscr{Q}$, we must have some $W$ such that $A_{n,W}$ is an uncountable antichain. In particular, $A_{n,W}$ has two distinct elements $V$ and $U$. BUT!!!

We have $\mu(U) \leq \epsilon - \frac{3}{n}$ and $\mu(U \Delta W) < \frac{1}{n}$ and $\mu(V \Delta W) \leq \frac{1}{n}$. We also have

$$U \cup V \subseteq U \cup (U \Delta W) \cup (V \Delta W),$$

and so

$$\begin{aligned} \mu(V \cup U) &\leq \mu\left(U \cup (U \Delta W) \cup (V \Delta W)\right) \\ &\leq \mu(U) + \mu(U \Delta W) + \mu(V \Delta W) \\ &\leq \epsilon - \frac{3}{n} + \frac{1}{n} + \frac{1}{n} \\ &= \epsilon - \frac{1}{n} \\ &< \epsilon \end{aligned}$$

Thus $U$ and $V$ are compatible, contradicting our statement that this is an antichain. $\square$

## 4.4 Homework!

Let's do some nasty analysis! For the purposes of these problems, assume that **all** intervals—open, closed, and half-open—behave nicely with respect to our measure function $\mu$. That is—the measure of any one of these intervals is equal to its length, and the countable union of disjoint intervals is equal to the sum of the measures of the intervals.

1. Show that for any open interval $(a, b) \subseteq \mathbb{R}$ and for any $\epsilon > 0$, there exists a rational interval $(q, r) \subseteq \mathbb{R}$ with $q, r \in \mathbb{Q}$ such that $\mu((a, b)\Delta(q, r)) < \epsilon$.

2. Show that for any open set $U$ and any $\epsilon > 0$, there exists an open set $W$ which is the union of countably many intervals with rational (or infinite) endpoints such that $\mu(U\Delta W) < \epsilon$.

3. Show that for any open set $U$ and any $\epsilon > 0$, there exists an open set $W$ which is the union of *finitely* many intervals with rational (or infinite) endpoints such that $\mu(U\Delta W) < \epsilon$.

4. Show that given countably many open sets $U_0, U_1, U_2, U_3, \ldots$, we have

$$\mu\left(\bigcup_{n=0}^{\infty} U_n\right) = \lim_{m\to\infty} \mu\left(\bigcup_{n=0}^{m} U_n\right)$$

# 5 Let's Kill This Off!

## 5.1 Our Poset Is CCC!

Last time we did a lightning-fast run through the proof that our poset of open sets is CCC... Let's do that again, but slower.[11]

**Theorem:** $\mathbb{P}$ is CCC.

*Proof.* Two sets $U$ and $V$ in $\mathbb{P}$ are incompatible if and only if $\mu(U \cup V) \geq \epsilon$. Let's assume that some uncountable antichain $\mathscr{A}$ exists. Let's define

$$\mathscr{A}_n = \left\{ U \in \mathscr{A} : \mu(U) \leq \epsilon - \frac{3}{n} \right\}.$$

Then we have

$$\mathscr{A} = \bigcup_{n=1}^{\infty} \mathscr{A}_n.$$

The countable union of countable sets is countable, and $\mathscr{A}$ is uncountable. Thus it must be the case that there exists some $n \in \mathbb{N}$ such that $\mathscr{A}_n$ is an uncountable antichain.

Let $\mathscr{Q}$ be the set of all open sets obtained by taking the union of finitely many disjoint open intervals with rational endpoints. We know[12] that for any open set $U$ and any $\epsilon > 0$, there exists a set $W \in \mathscr{Q}$ such that $\mu(U \Delta W) < \epsilon$.

We now define

$$\mathscr{A}_{n,W} = \left\{ U \in \mathscr{A}_n : \mu(U \Delta W) \leq \frac{1}{n} \right\}.$$

The result stated above gives us

$$\mathscr{A}_n = \bigcup_{W \in \mathscr{Q}} \mathscr{A}_{n,W},$$

and since $\mathscr{A}_n$ is uncountable and ther eare only countably many $W \in \mathscr{Q}$, we must have some $W$ such that $\mathscr{A}_{n,W}$ is an uncountable antichain. In particular, $\mathscr{A}_{n,W}$ has two distinct elements $V$ and $U$. BUT!!!

[11]. . . and maybe without all the typos I put in those proofs yesterday.

[12]From the homework if we did it, or by Appeal To Authority if we didn't.

We have $\mu(U) \leq \epsilon - \frac{3}{n}$ and $\mu(U\Delta W) < \frac{1}{n}$ and $\mu(V\Delta W) \leq \frac{1}{n}$. We also have

$$U \cup V \subseteq U \cup (U\Delta W) \cup (V\Delta W),$$

and so

$$\begin{aligned}
\mu(V \cup U) &\leq \mu\left(U \cup (U\Delta W) \cup (V\Delta W)\right) \\
&\leq \mu(U) + \mu(U\Delta W) + \mu(V\Delta W) \\
&\leq \epsilon - \frac{3}{n} + \frac{1}{n} + \frac{1}{n} \\
&= \epsilon - \frac{1}{n} \\
&< \epsilon
\end{aligned}$$

Thus $U$ and $V$ are compatible, contradicting our statement that this is an antichain. $\square$

## 5.2 But Is It Too Big?

Thus our poset $\mathbb{P}$ has a filter $G$ which intersects with all sets

$$\{U \in \mathbb{P} : S_\alpha \subseteq U\}.$$

If we take the set

$$\bigcup_{U \in G} U,$$

Then clearly we have an open set which contains all the $S_\alpha$ as subsets. But is $U$ small enough? To show that it is, again we'll consider open sets with rational endpoints. But this time we won't need anything as complicated as $\mathscr{Q}$. All we'll need is intervals with rational endpoints.

**Lemma:**

$$\bigcup_{U \in G} U = \bigcup_{I \in (G \cap \mathscr{Q})} I$$

*Proof.* It's clear that $\bigcup_{I \in (G \cap \mathscr{Q})} I \subseteq \bigcup_{U \in G} U$, so let's focus on the other containment.

Suppose $x \in \bigcup_{U \in G} U$. Then there is some $U \in G$ such that $x \in U$. Since $U$ is open, there exists $\delta > 0$ such that $(x - \delta, x + \delta) \subseteq U$. There exist rational numbers $q$ and $r$ such that $q \in (x - \delta, x)$ and $r \in (x, x + \delta)$. Thus we have

$$x \in (q, r) \subseteq U.$$

Since $G$ is closed upwards, we have $(q, r) \in G$, and so $(q, r)$ is one of the intervals in $(G \cap \mathscr{Q})$. Thus we have

$$x \in \bigcup_{I \in (G \cap \mathscr{Q})} I$$

$\square$

So! Let's use this to prove that $\mu\left(\bigcup_{U \in G} U\right) \leq \epsilon$. We have

$$\bigcup_{U \in G} U = \bigcup_{I \in (G \cap \mathscr{Q})} I,$$

and we know that there are only countably many intervals $I$ in $G \cap \mathscr{Q}$. Thus we can enumerate these intervals to get

$$\begin{aligned} \mu\left(\bigcup_{U \in G} U\right) &= \mu\left(\bigcup_{n=1}^{\infty} I_n\right) \\ &= \lim_{m \to \infty} \mu\left(\bigcup_{n=1}^{m} I_n\right). \end{aligned}$$

Since elements of $G$ are pairwise compatible, we have

$$\mu\left(\bigcup_{n=1}^{m} I_n\right) < \epsilon$$

for every $m$, and so we have

$$\lim_{m \to \infty} \mu\left(\bigcup_{n=1}^{m} I_n\right) \leq \epsilon$$