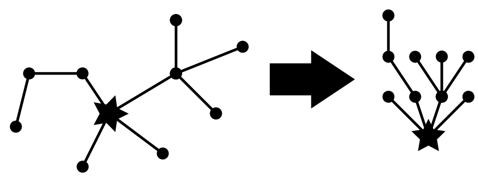
1 Infinite Trees: Day 1

1.1 So What's A Tree?

First, let's define what we mean intuitively when we talk about a "tree". Graph theorists will tell you that a tree is any acyclic graph.[1] But that's not quite enough for our purposes.[2] We're really going to want to be able to talk about the notion of height in this class. So what we'll do is pick an element $\star$ of the vertex set of this graph and call it the **root**. We can imagine grabbing the root and pulling downwards until all of the other vertices fall into place making something that actually looks tree-like, as shown below:



This gives us the ability to measure the **height** of various elements of our tree. The height of the root is zero—we write $\mathrm{ht}(\star) = 0$. The height of the elements that are connected to $\star$ is 1. The height of the elements connected to those elements is 2, and so on. For a number n, we use the notation T_n for the collection of all elements with height n. We call this the n^{th} **level** of T.

1.2 König's Infinity Lemma

König's Infinity Lemma states that if T is a tree of infinite height, and all T_n are finite, then T must have an infinite branch.

To see this, let b_0 be the root of the tree. Since there are infinitely many vertices in the tree, and only a finite number of vertices in T_1, there must be some vertex in T_1 that has infinitely many vertices above it by the infinite pigeonhole principle. Call this vertex b_1.

To construct the n^{th} vertex, assume that we already have a branch up to b_{n-1}. Since each level is finite, b_{n-1} can have only finitely many nodes above it in T_n. Since b_{n-1} has infinitely many vertices above it, there must be some vertex

[1] Well, any *connected* acyclic graph. Technically the objects we're focusing on in this class are what graph theorists would refer to as forests.

[2] It also doesn't really give you anything that looks recognizably like a tree.

above b_{n-1} in T_n with infinitely many vertices above it. Call this vertex b_n. Since we can find a b_n in each level of T, we can construct an infinite branch.[3]

König's infinite lemma deals with infinite graphs which we implicitly assume to be countable. But what happens if we try to extend the statement so that it works one cardinality up?

Extension: If T is a tree of uncountable height, and each level T_α is countable, then T must have an infinite branch.

Before we can talk about whether this extension is true, we're going to have to make sense of this notion of a tree of uncountable height. For this, we'll need the ordinals.

1.3 How To Count The Uncountable

begin{Disclaimer} This is not a formal development of the ordinal numbers. A lot of beautiful mathematics is being swept under the rug, since all we really need to know is how to count up to an uncountable number. **end{Disclaimer}**

So: how do we count to an uncountable number? We start in the usual way, with 0, 1, 2, 3, etc.[4] We keep going until we've counted all the natural numbers. Then we count ω. And then $\omega + 1$, $\omega + 2$, $\omega + 3$, and so on until we've run out of those, and then we count $\omega 2$.[5] We can keep going and get to $\omega 3$, $\omega 4$, $\omega 5$, and so on, and once we've counted all of those we get ω^2. Then we can get ω^3, ω^4, and eventually ω^ω. Then ω^ω, ω^{ω^ω}, $\omega^{\omega^{\omega^\omega}}$, and eventually all these exponents pull the thing sideways and we get ϵ_0, because clearly epsilon should be a really really big number. Of course, it's not really all that big, because all of the numbers we can get to by doing this sort of thing are countable anyway.[6]

So let's do this a little more formally. When we study set theory, we would like to consider all mathematical objects as sets. The natural numbers is a collection of sets that "counts" things. So if we say that a set "has 3 elements," what we mean is that my set has the same number of elements as the set that

[3] Modulo Zorn's lemma. But as far as we're concerned in this class, the axiom of choice is true.

[4] I always start counting with zero. I get more cookies that way.

[5] Not 2ω. That means something else. See disclaimer.

[6] Exercise!

we have decided to call "3". It turns out that an easy way to do this is to define each natural number to be the set of all smaller natural numbers.

Since there is no natural number smaller than zero, we'll have $0 = \emptyset$. Since the only natural number smaller than one is zero, we have $1 = \{0\} = \{\emptyset\}$. We continue in like fashion and obtain

$$\begin{aligned} 2 &= \{0,1\} &&= \{\emptyset, \{\emptyset\}\}, \\ 3 &= \{0,1,2\} &&= \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \end{aligned}$$

and so on. Not only does this give us sets that "count" in the way described above. It also creates a nice correspondence between the order relation $<$ on the natural numbers, and the is-an-element-of relation $\in$ in Set Theory Land.

If we get to the end of the natural numbers and want to keep counting, we can define ω to be the smallest number that is larger than all the natural numbers. Thus

$$\omega = \{0, 1, 2, 3, \ldots\}$$

By the same token, we can define ω_1 to be the set of all countable ordinals. For any countable ordinal α, we have $\alpha \in \omega_1$. Since no set can be an element of itself, it follows that ω_1 must be uncountable.

1.4 Some Definitions We Will Need

Given any ordinal number α, we define $\alpha+1$, the **successor** of α, to be $\alpha \cup \{\alpha\}$. If an ordinal β is equal to $\alpha+1$ for some ordinal α, then we call β a **successor ordinal**. Otherwise, β is a **limit ordinal**.

Just as the ordinal numbers count finite sets, the ordinal numbers count **well-ordered** sets. A poset $(X, <)$ is **well-ordered** if for any subset $S \subseteq X$ there exists $m \in S$ such that $m \leq s$ for any $s \in S$. That is, every subset of X has a minimum with respect to the relation $<$.

To "count" a well-ordered set, we start by matching the smallest element with 0. Then we match the smallest remaining element with 1. We keep going until we've used all the natural numbers. Then we match the smallest remaining element with ω, and keep on going. In this way, we can build an order-isomorphism between our well-ordered set and a unique ordinal. The **order type** of a well-ordered set S is the unique ordinal that is order-isomorphic to the set—the ordinal that "counts" it.

We'll need to be a little careful here, because the order type will *not* be the largest ordinal that gets "matched" in this way. If the largest matched ordinal is α, then the order type will be $\alpha + 1$. If there is no largest ordinal matched, then the order type is a limit ordinal.

Now that we understand what it means to count uncountable sets, let's take a trip to Formality Land so that we can see what an infinite tree might look like!

1.5 Formality Land!

A **tree** is a poset $(T, <)$ with a unique minimal element, such that for any $x \in T$, the set $\mathrm{pred}(x)$ of predecessors of x, given by

$$\mathrm{pred}(x) = \{y \in T : y < x\},$$

is well-ordered by $<$. Notice that this captures the intuitive notion that from any node in a tree, we can trace a unique path down to the root.

The **height** of a node $x \in T$, denoted by $\mathrm{ht}_T(x)$, is the order type of $\mathrm{pred}(x)$. For an ordinal number α, the α^{th} **level** of T, denoted T_α, is given by

$$T_\alpha = \{x \in T : \mathrm{ht}_T(x) = \alpha\}.$$

The **height of** T, denoted $\mathrm{ht}(T)$, is given by

$$\mathrm{ht}(T) = \min\{\alpha : T_\alpha = \emptyset\}$$

This is the first ordinal number whose corresponding level is empty.

We define a **branch** to be a maximal totally ordered subset of T. That is, a totally ordered set $B \subseteq T$ such that adding any other element of T results in a non-totally-ordered set.

We define a subset S of a tree T to be a **subtree** of T if it contains the uniqe minimal element and forms a tree under the T-relation. It is easy to check that this will be the case if and only if for all $t \in S$ we have $\mathrm{pred}(t) \subseteq S$. Notice that a branch $B \subseteq T$ is always a subtree of T.

For an ordinal number α, we will use the notation

$$T^\alpha = \{t \in T : \mathrm{ht}(t) < \alpha\}.$$

This is another easy example of a subtree.

Example: The set ω_1 with the relation $\in$ is a really boring tree. The height of an ordinal $\alpha \in \omega_1$ is α. Any countable ordinal α is a subtree of the tree ω_1.

Less Stupid Example: Let T be the set of functions from any natural number $n \in \omega$ to the set $2 = \{0, 1\}$. We define $f < g$ if g is an **extension** of f. That is to say, $f : n \to 2$ and $g : m \to 2$ and for any $p \in n$, we have $f(p) = g(p)$.

1.6 Now Back To Our Original Statement...

...which turns out to be completely false. This brings us to our final definition of the day! An **Aronszajn tree** is a tree T satisfying the following three properties:

(i) $\mathrm{ht}(T) = \omega_1$.

(ii) $|T_\alpha| < \omega_1$ for all $\alpha < \omega_1$.

(iii) T has no uncountable branches.

Our goal in the next few days is to prove that these things actually exist. Stay tuned!

1.7 Homework!

1. Let A be an unbounded set of ordinal numbers: that is, for any $\alpha \in A$, there exists $\alpha' \in A$ with $\alpha' > \alpha$. Let β be the smallest ordinal such that $\beta > \alpha$ for all $\alpha \in A$. Prove that $\beta = \bigcup_{\alpha \in A} \alpha$. We call β the **limit** of the set A.[7]

2. Prove that any ordinal obtained by schwooooomping[8] a finite number of times is countable.

3. **König's Infinity Lemma (Awesome Version):** If G is a connected graph with infinitely many vertices, and each vertex has finite degree, show that G contains an infinite path.

4. **Cayley's Formula (Stupid Version):** Prove that the number of labeled trees on a countably infinite set of vertices is uncountable.

[7] This is also true if A is a bounded set of ordinals. It's just a lot dumber.

[8] Totally a word.

2 Infinite Trees Day 2

2.1 Posets of Functions

Quite often, our trees will take the form of partially ordered sets of functions. For example, suppose we take T to be the collection of all functions

$$f : n \to 2,$$

where n is allowed to range across all elements of ω. We order this set by saying $f < g$ if and only if g is an extension of f. Clearly the predecessors of a given function in T form a well-ordered set. This tree will have countably many nodes, and uncountably many branches.[9]

2.2 Aronszajn Trees

Now recall the statement made in the last class:

Statement: If T is a tree of uncountable height, and each level T_α is countable, then T must have an uncountable branch.

This statement turns out not to be true. In fact, we will begin the construction of a counterexample today!

Definition: We call a tree T an **Aronszajn tree** if

(i) $\text{ht}(T) = \aleph_1$

(ii) $|T_\alpha| < \aleph_1$ for all $\alpha < \omega_1$.

(iii) Every branch $B \subseteq T$ is countable.

As a first approximation to a construction, let's build a tree that satisfies the first and third conditions: Let T be the collection of all functions $t : \alpha \to \omega$ satisfying

(a) $\alpha < \omega_1$

(b) t is injective

(c) $|\omega - \text{im}(t)| = \aleph_0$

[9] Actually, this is the infinite binary tree for those of you who know of such things.

We define an ordering $<$ on T by setting $t_1 < t_2$ if t_2 is an extension of t_1. This means that any function $t \in T_{\alpha+1}$ is defined on $1, 2, 3, \ldots, \alpha$. A function $t \in T_\beta$ for a limit ordinal β is defined on all $\alpha < \beta$. We wish to show that our tree satisfies properties (i) and (iii).

Our Tree Satisfies (i): Since nodes exist on every level of this tree below ω_1, the height of the tree is ω_1. $\square$

Our Tree Satisfies (iii): Suppose there exists an uncountable branch $B = \{t_\alpha : \alpha < \omega_1\}$, with each $t_\alpha : \alpha \to \omega$. Then we can use B to define a function $f : \omega_1 \to \omega$ as follows: for each $\alpha < \omega_1$, we define $f(\alpha) = t_{\alpha+1}(\alpha)$. We claim that this is an injective function.

To see that this is the case, suppose $f(\alpha) = f(\beta)$ for some $\alpha, \beta < \omega_1$. Let $\gamma = \max\{\alpha+1, \beta+1\}$. Then $t_\gamma(\alpha) = t_\gamma(\beta)$, and since t_γ is injective, it follows that $\alpha = \beta$.

Problem is, this gives us an injective function $f : \omega_1 \to \omega$, which implies that $|\omega_1| \leq |\omega|$. This means that the uncountable branch B that we used to build f can't possibly exist. $\square$

2.3 Our Tree Is Too Fat!

This is cool and all, but if we look at the size of the levels in our tree, we find that the ω^{th} level, the injective functions from ω to ω avoiding an infinite number of elements in the domain, is uncountable.

To see this, consider the set S of functions from ω to ω avoiding the odd numbers. This is contained in the set P that avoids infinitely many elements of the domain. However, the functions in S can be put into one-to-one correspondence with the injective functions from ω to ω, by matching $f \in S$ with $g = \frac{1}{2}f$. Thus the number of elements of P is equal to the number of injective functions from ω to ω, which is uncountably large. So our tree is too fat to be an Aronszajn tree.

2.4 Okay, Let's Build An Aronszajn Tree!

We'd like to find a way to "thin out" the tree we've just created to get an Aronszajn tree. To obtain an Aronszajn subtree T^* of our tree T, we start by defining an equivalence relation on the set of functions. If s and t are functions from α to ω, we'll say $s \sim t$ if the are *almost* equal to each other.

$$|\{\beta < \alpha : s(\beta) \neq t(\beta)\}| < \omega$$

Notice[10] that if $\alpha < \omega_1$ and $t : \alpha \to \omega$, then there are only countably many $s : \alpha \to \omega$ with $s \sim t$.

So we choose one $s_\alpha \in T_\alpha$ for every $\alpha < \omega_1$, and we define T^*_α to be the equivalence class of s_α. Now, we might run into a problem, because unless we choose our s_α carefully, there is no guarantee that $\text{pred}(t)$ will be in our tree for every $t \in T^*$. In order for our construction to work out, every element of the equivalence class of s_α has to be connected to (greater than) some element of the equivalence class of s_β for every $\beta < \alpha$. Thus the condition we're looking for is:

Condition: If $\beta < \alpha < \omega_1$ and $t \sim s_\alpha$, then $t|_\beta \sim s_\beta$.

We claim that this condition holds if and only if $(s_\alpha)|_\beta \sim s_\beta$. The "only if" direction is clear. To prove the "if" direction, take some $t \sim s_\alpha$. Obviously, $t|_\beta \sim (s_\alpha)|_\beta$, since the restrictions of t and s_α to β are going to differ at strictly fewer points than t and s_α. By assumption, we have $(s_\alpha)|_\beta \sim s_\beta$. Since this is an equivalence relation, it follows that $t|_\beta \sim s_\beta$.

So all that remains in order to prove the existence of our Aronszajn tree is to show the following:

Claim: There exist functions s_α for all $\alpha < \omega_1$ such that the following conditions hold for each α:

(1) $s_\alpha : \alpha \to \omega$.

(2) s_α is injective.

(3) $|\omega - \text{im}(s_\alpha)| = \omega$.

(4) For any $\beta < \alpha < \omega_1$, we have $(s_\alpha)|_\beta \sim s_\beta$.

To prove this claim, we're going to need a tool called transfinite induction.

2.5 Transfinite Induction

We can prove that something is true for all the natural numbers by using induction. There is a similar process called transfinite induction that allows

[10] Exercise!

us to prove that something is true for all the ordinal numbers. Here's how it works:

Base Case: Prove that your statement is true for 0.

Successor Case: Prove that if your statement is true for an ordinal α, then it is also true for $\alpha + 1$.

Limit Case:[11] Prove that if γ is a limit ordinal and your statement is true for every $\alpha < \gamma$, then your statement is also true for γ.

2.6 Sweet, Lets See It In Action!

We start with your base case. We need a function s_0 from the empty set to ω. Pick the empty function. Done.

For the successor case, assume that s_α has been defined and satisfies (1)-(4). Define $s_{\alpha+1}(\beta) = s_\alpha(\beta)$ for all $\beta < \alpha$, and choose an arbitrary $n \notin \mathrm{im}(s_\alpha)$ and set $s_{\alpha+1} = n$.

This leaves the limit case. This is usually the hardest part of a transfinite induction argument. Let $\gamma < \omega_1$ be a limit ordinal, and suppose that s_α has been defined and satisfies (1)-(4) for all $\alpha < \gamma$.

Just for example, let's try to construct s_ω. Up until s_ω, all of these functions agree with each other. So suppose we try to take the union of all these functions. Well, we could get unlucky. If for each n we define s_n to be the function sending m to itself for every $m < n$, then the union of all these functions will be the identity function on ω, which will not satisfy condition (3), even though each s_n does. So we need to be cleverer. We'll see just how much cleverer we need to be next time!

[11] Also known as "Shvooooomp!"

2.7 Homework!

1. In class we said that the equivalence class of any s_α is countable. Let's prove it!
   (a) Find an injective map from the equivalence class of s_α into the set of finite strings of natural numbers.
   (b) Find an injective map from the finite strings of natural numbers to the finite strings of zeroes and ones.
   (c) Find an injective map from the finite strings of zeroes and ones into the natural numbers.
2. The crucial point in today's lecture is that any $\{s_\alpha : \alpha < \omega_1\}$ satisfying the **Condition** in section **??** will give us an Aronszajn tree. Make sure that you believe that this is true. Tomorrow we'll be building such a sequence!
3. Prove that given a limit ordinal $\gamma < \omega_1$, there exists an increasing sequence of ω ordinals $\alpha_1 < \alpha_2 < \alpha_3 < \ldots < \alpha_n < \ldots$ such that $\bigcup_{i \in \omega} \alpha_i = \gamma$.
4. A tree is said to be **well pruned** if for any $\alpha < \beta < \omega_1$ and $x \in T_\alpha$, there exists $y \in T_\beta$ such that $x < y$.
   (a) Find an example of a well-pruned tree of height ω_1.
   (b) Do you think there could exist a well-pruned Aronszajn tree?

3 Infinite Trees Day 3

3.1 Okay Seriously, Let's Build An Aronszajn Tree

Recal that we are in the process of constructing an Aronszajn tree. We started with the tree T consisting of the set of all functions $t : \alpha \to \omega$ for some $\alpha < \omega_1$ satisfying

(i) t injective.

(ii) $(\omega \setminus \mathrm{im}(t))$ infinite.

We ordered T by extension, and found that $\mathrm{ht}(T) = \omega_1$, and that T has no uncountable branch. But T *does* have uncountable levels. So we're trying to find a subtree $T^* \subseteq T$ so that T^*_α is the equivalence class of some $s_\alpha \in T_\alpha$ under the almost-equals relation.

This will certainly have no infinite branch—we can't accidentally create one just by deleting stuff. And by design each level will be countable, since the set of functions from α to ω that differ from a particular s_α at only finitely many places is a countable set. However, we may run into some issues if the subset of T that we choose turns out not to be a tree.

In order for our structure to be a tree, any $t \in T_\alpha$ must have some element below it in T_β for all $\beta < \alpha$. That is, for any $t \sim s_\alpha$, we need $t|_\beta \sim s_\beta$. In fact, it will be good enough to show that for any $\beta < \alpha < \omega_1$, we have $s_\alpha|_\beta \sim s_\beta$. If this is true, then for any $t \in T_\alpha$, we have $t \sim s_\alpha$, and so $t \sim s_\alpha|_\beta \sim s_\beta$.

So then what we need is a collection of functions $\{s_\alpha : \alpha < \omega_1\}$ satisfying

(a) $s_\alpha : \alpha \to \omega$ for all $\alpha < \omega_1$.

(b) $(\omega \setminus \mathrm{im}(s_\alpha))$ infinite for all $\alpha < \omega_1$.

(c) s_α injective for all $\alpha < \omega_1$.

(d) $s_\alpha|_\beta \sim s_\beta$ for all $\beta < \alpha < \omega_1$.

We are attempting to construct such a collection of functions using transfinite recursion.[12] For the base case, we set $s_0 = \emptyset$.

For the successor case, if we have defined functions for all ordinals less than or equal to α, we define $s_{\alpha+1}$ to be equal to s_α on all ordinals less than α,

[12] You prove things using transfinite induction. You build things using transfinite recursion!

and we set $s_{\alpha+1}(\alpha)$ equal to…eh…anything that hasn't been used yet. This function will certainly satisfy (a), (b), and (c). To show that it satisfies (d), notice that for any $\beta < (\alpha + 1)$, $s_{\alpha+1}|_\beta = s_\alpha|_\beta$. Since $s_\alpha|_\beta \sim s_\beta$ for all $\beta \leq \alpha$, we have $s_{\alpha+1}|_\beta \sim s_\beta$. That's it for the successor step. This brings us to…

3.2 The Limit Case!

Here's the idea: for all $\alpha < \omega_1$, we need $(\omega \setminus \mathrm{im}(s_\alpha))$ to be infinite. So for a limit ordinal γ, we can't just extend the previous functions the way we did in the successor case, because we may fill up ω too fast and violate condition (b). So we need to find a way to construct s_γ while simultaneously setting aside $\aleph_0$ elements in ω that we will never use. This will take some awkward shuffling around to make sure we don't break condition (d) in the process. We'll need the following lemma:

Lemma: If γ is a countable limit ordinal, there exists an ω-sequence $\alpha_0, \alpha_1, \alpha_2, \ldots,$ such that $\bigcup_{n\in\omega} \alpha_n = \gamma$.

Proof. There exists a bijection between γ and ω. Look at the largest strictly-increasing subsequence

$$\alpha_{i_0} < \alpha_{i_1} < \alpha_{i_2} < \ldots$$

This subsequence has order type ω. Switching the subscripts to the natural numbers gives us the sequence we want. □

We'll use this sequence of ordinals to build a sequence of functions $t_0 < t_1 < t_2 < \ldots \in T$ such that $t_i : \alpha_i \to \omega$ for all i, and that if $i > j$, then t_i is an extension of t_j. Then taking the union of the t's will give us a function $s_\gamma : \gamma \to \omega$. During this construction, for each t_i we want to "set aside" a value m_i and promise ourselves that we won't let it end up in the image of the function s_γ, so that at the end s_γ will satisfy (b).

This will be an inductive construction inside a recursion construction! We need each t_i to be an injective function satisfying $t_i \sim s_{\alpha_i}$, and to be an extension of all the t_j with $j < i$. For our base case, we take $t_0 = s_{\alpha_0}$. Then since $(\omega \setminus s_{\alpha_0})$ is infinite,[13] we can pick an element m_0 from this set. The smallest one, if you want. We agree that from now on, we will not allow any of the t_i to use m_0 as one of their values.

[13]In particular, it is nonempty.

For our inductive step, we assume we have a function t_i satisfying $t_i \sim s_{\alpha_i}$, and special elements $m_0, m_1, \ldots, m_i$ that are not in the image of t_i.We define t_{i+1} by setting $t_{i+1}(\beta) = s_{\alpha_{i+1}}(\beta)$, with the following exceptions:

(1) If t_i sent β somewhere else, then set $t_{i+1}(\beta) = t_i(\beta)$.

(2) If $s_{\alpha_{i+1}}(\beta) \in \{m_0, m_1, \ldots, m_i\}$, then set t_{i+1} to something else that's still available.

(3) If $s_{\alpha_{i+1}}(\beta) = t_i(\delta)$ for $\delta < \beta$, then set t_{i+1} to something else that's still available.

This is clearly an extension of t_i, by (1). This is clearly injective, by (3). But how often does t_i differ from s_{α_i}? Do we have $t_i \sim s_{\alpha_i}$? We have

$$s_{\alpha_{i+1}}|_{\alpha_i} \sim s_{\alpha_i} \sim t_i,$$

so (1) can happen only finitely many times. Since s_{α_1} is injective, (2) can only happen i times.[14]. Condition (3) can only occur where t_i is not equal to $s_{\alpha_{i+1}}|_{\alpha_i}$. Since, as we just mentioned,

$$t_i \sim s_{\alpha_{i+1}}|_{\alpha_i},$$

this only happens finitely many times. This means that t_{i+1} can only differ from $s_{\alpha_{i+1}}$ in finitely many places. This gives us $t_{i+1} \sim s_{\alpha_{i+1}}$. Now we pick a new element $m_{i+1} \in (\omega \setminus \mathrm{im}(s_{\alpha_{i+1}}))$ and promise not to use it in the future.

Now that we have this sequence of functions, we take their union to get s_γ. By construction, we know that s_γ satisfies conditions (a) and (b). To show that s_γ is injective, suppose $s_\gamma(\beta) = s_\gamma(\delta)$. There exists α_i with $\alpha_i > \beta$ and $\alpha_i > \delta$. We have $s_\gamma|_{\alpha_i} = s_{\alpha_i}$, so $s_{\alpha_i}(\beta) = s_{\alpha_i}(\delta)$, and thus $\beta = \delta$.

All that remains is to show that for any $\beta < \gamma$, we have $s_\gamma|_\beta \sim s_\beta$. Pick some α_n in γ such that $\alpha_n > \beta$. We have

$$s_\gamma|_{\alpha_n} = t_n \sim s_{\alpha_n}.$$

Restricting further doesn't break the $\sim$ relation, so this gives us

$$s_\gamma|_\beta \sim s_{\alpha_n}|_\beta \sim s_beta.$$

This gives us our Aronszajn tree! Hooray!!!

[14]In particular, i is a finite number!

3.3 Coming Up Soon!

Recall that an Aronszajn tree is **well pruned** if for any $\alpha < \beta < \omega_1$ and for any $t \in T_\alpha$, there exists $s \in T_\beta$ such that $t < s$. In a well-pruned tree we can start at any vertex and "jump up" to a vertex at any level we choose. It seems like this would not be compatible with the no-uncountable-branches condition in our Aronszajn tree. But in fact, we have the following:

Fact 1: There exists a well-pruned Aronszajn tree.

But we can do even better than that!

Fact 2: Every Aronszajn tree has a well-pruned Aronszajn subtree.

Totally mind-blowing! Totally awesome!! Totally proving it next week!!![15]

[15]Exclamation marks are used here to communciate excitement, not to denote factorials.

3.4 Homework

1. The Aronszajn tree that we constructed in class today is actually well-pruned. Can you prove it?

2. An Aronszajn tree is **special** if it is the union of a countable number of antichains. Can you construct a special Aronszajn tree?[16]

3. We call a tree $\mathbb{Q}$-embeddable if there exists a (not necessarily injective) function $f : T \to \mathbb{Q}$ such that $f(t) < f(s)$ whenever $t < s$. Show that an Aronszajn tree is special if and only if it is $\mathbb{Q}$-embeddable.

[16]This is a tough problem. Ask Susan if you want the definition of an antichain.

4 Infinite Trees Day 4

4.1 Well-Pruned-Ness

Recall that a tree of height ω_1 is **well-pruned** if whenever $t \in T_\alpha$, and $\alpha < \beta < \omega_1$, there exists $s \in T_\beta$ such that $t < s$. That is, if we pick an arbitrary node t in T_α, and an arbitrarily large ordinal $\alpha < \beta < \omega_1$, we can find a path from t to some node s in T_β.

Stupid Example: The countable ordinals form a well-pruned tree under the $\in$ relation.

But what about Aronszajn trees? If an Aronszajn tree is well-pruned, then we have an uncountably tall tree with no path going all the way up. But in spite of this, we can get arbitrarily high up in the tree from any node we choose! Nonetheless these things exist. In fact, we have the following:

Theorem: Any Aronszajn tree has a well-pruned Aronszajn subtree.

Proof. When we take a subtree of an Aronszajn tree, we do this by throwing things away, so we don't have to worry about creating an uncountable branch by mistake. So as long as we make sure our subtree has height ω_1, it is guaranteed to be Aronszajn. This means that we want to find a subtree T' such that for any $\alpha < \beta < \omega_1$, and for any $t \in T_\alpha$, there exists $s \in T_\beta$ with $t < s$.[17]

We will define our subtree T' as follows:

$$T' = \{s \in T : |\mathrm{succ}(s)| = \aleph_1\}.$$

Notice that if $t < s$, then $\mathrm{succ}(t) \subseteq \mathrm{succ}(s)$, so $|\mathrm{succ}(s)| = \aleph_1$ implies that $|\mathrm{succ}(t)| = \aleph_1$. This means that whenever $s \in T'$, we also have $\mathrm{pred}(s) \subseteq T'$, and so T' is a subtree of T.

To show that T' has height ω_1, we must show that for any $\alpha < \omega_1$, there exists some $t \in T'_\alpha$. Notice that T_α has only countably many nodes, but since T is uncountably tall, there must be uncountably many nodes somewhere above the α^{th} level. Thus by the uncountable pigeonhole principle, there must be some $t \in T_\alpha$ with uncountably many successors. This t will be an element of T'_α.

[17]Notice that this will guarantee that the height of our tree is ω_1.

Now we wish to show that for any $\alpha < \beta < \omega_1$ and for any $t \in T'_\alpha$, there exists $s \in T'_\beta$ such that $t < s$. We know that t has uncountably many successors in T. We also know that since a countable union of countable sets is countable, $T^{\beta+1}$ is countable. This means that there exist at most countably many nodes above t at or below the level T_β. This means that s has countably many successors in T_β, and uncountably many successors above T_β. This means that one of its successors in T_β must have uncountably many successors. Thus there must exist $t \in T'_\beta$ such that $t > s$. $\square$

It is interesting to note that the Aronszajn tree we constructed is already well-pruned. If $\alpha < \beta > \omega_1$, and t is on the α^{th} level, we have

$$t \sim s_\alpha \sim s_\beta|_\alpha.$$

If we define a function $t_* : \beta \to \omega$ so that $t_*(\gamma) = t(\gamma)$ for all $\gamma < \alpha$, and otherwise send the ordinals to the same place as s_β "wherever possible," we should be able to construct a function that is "almost equal" to s_β. Let's do this as an exercise!

4.2 Suslin Trees

In spite of the fact that a well-pruned Aronszajn tree is a fairly mind-blowing concept, it turns out that it isn't a particularly interesting strengthening. We can cut down any Aronszajn tree to make it into a well-pruned tree. So let's look at something more interesting.

Given a tree T, we define a subset $A \subseteq T$ to be an **antichain** if for any $a, b \in A$, we have $a < b$ and $b < a$. For example, any level is clearly an antichain. Now, we've constructed an ω_1-tree with no uncountable branches and no uncountable levels. What if we wanted to construct an ω_1-tree with no uncountable branches and no uncountable antichains?

Definition: We call T a Suslin tree if

(i) $\mathrm{ht}(T) = \omega_1$

(ii) T has no uncountable branches.

(iii) T has no uncountable antichains.

Of course, we can define any thing we want—the question is: does it exist? and the answer is a resounding "Maybe!" It turns out that the axioms of ZFC are not sufficiently strong to either construct a Suslin tree or to prove that no such thing exists. But for now let's remain optimistic and see what we can learn from a couple of failed attempts at constructing one.

4.3 Attempt I: Maybe We Got Lucky

Okay, so any Suslin tree is an Aronszajn tree. Maybe we got lucky and constructed an Aronszajn tree on our first try. That would be super cool. Unfortunately, we have the following:

Theorem: Let T be any tree of height ω_1 such that for all $\alpha < \omega_1$

$$T_\alpha \subseteq \{t : \alpha \to \omega \mid t \text{ injective}\}.$$

Then T is not a Suslin tree.

Proof. For each $n \in \omega$, define a set

$$A_n = \{t \in T : \exists\ \alpha \text{ s.t. } \mathrm{dom}(t) = (\alpha + 1) \text{ and } t(\alpha) = n\}.$$

That is, A_n is the collection of functions whose largest defined value is n. This set is an antichain since any two elements of A_n are either on the same level, in which case they are incomparable, or they disagree about which ordinal to send to n, in which case they are incomparable.

For each $\alpha < \omega_1$, choose $t_\alpha \in T$ such that $t : (\alpha + 1) \to \omega$. We know that $t_\alpha \in T_{\alpha+1}$. We know that the set $\{t_\alpha : \alpha < \omega_1\}$ is uncountable, and that each element is an element of one of the A_n. Since there are only countably many A_n, one of these antichains must be uncountable by the uncountable pigeonhole principle. $\square$

4.4 The Everbranching Condition

Okay, so maybe we were a little too greedy, hoping that our Aronszajn tree would magically turn out to be Suslin as well. But if we can't use injective functions, we're going to have to figure out some other way of preventing uncountable branches.

Definition: A tree $(T, <)$ is **everbranching** if for all $t \in T$, the set $\mathrm{succ}(t)$ is not linearly ordered.

If you've thought long and hard about how on earth a well-pruned Aronszajn tree could exist,[18] then you've probably come up with something that looks an awful lot like this everbranching condition. Follow a single branch and it runs out, but at some point along the way, you must have passed a place where you could have gone a different direction and gone higher. In fact, it must have been possible to go a different way and get higher from any point along your path.

The surprising thing about this everbranching condition, is that if we combine it with the height and antichain restrictions, it gives us the no-uncountable-branches condition for free.

Lemma: Suppose T is a tree such that

(a) T is ever-branching.

(b) $\mathrm{ht}(T) = \omega_1$.

(c) Every maximal antichain of T is countable.

Then T is a Suslin tree.

Proof. Clearly conditions (b) and (c) take care of the height and the antichains. We need to show that T will have no uncountable branches. So! Suppose B is an uncountable branch.

The everbranching condition guarantees that for every $b \in B$, there exists $t \in (T \setminus B)$ such that $b < t$. We will use this fact to create an uncountable antichain by transfinite recursion.

We begin by choosing some $b_0 \in B$, and a corresponding $t_0 \in (T \setminus B)$ with $b_0 < t_0$.

If we already have b_α and t_α, we pick $b_{\alpha+1}$ to be some element of B such that $\mathrm{ht}(b_{\alpha+1}) > \mathrm{ht}(t_\alpha)$, and take $t_{\alpha+1} \in (T \setminus B)$ such that $t_{\alpha+1} > b_{\alpha+1}$.

For a limit ordinal γ, if we have b_α and t_α for all $\alpha < \gamma$, we take b_γ to be a node in B that has height greater than

$$\sup_{\alpha<\gamma} \{\mathrm{ht}(t_\alpha)\}.$$

Since the supremum of a countable sequence of countable ordinals is countable, there must be some such b_γ. We take $t_\gamma \in (T \setminus B)$ such that $t_\gamma > b_\gamma$.

[18] And I know you have.

We claim that $\{t_\alpha : \alpha < \omega_1\}$ is an antichain in T. To see that this is the case, consider t_α and t_β for $\alpha < \beta$. Obviously, $t_\beta \not< t_\alpha$. We also know that $t_\beta > b_\beta$, and that $b_\beta \not> t_\alpha$. It follows that $t_\beta \not> t_\alpha$. $\square$

This may not seem like progress, but it actually is. It is fairly difficult to put together a construction to *avoid* something happening. Before, our task was to construct a tree with *no* uncountable branches and *no* uncountable antichains. Now we've replaced the no-uncountable-branch condition with a positive condition: our tree must be everbranching. If we can construct an everbranching tree, we won't have to do any work to keep uncountable branches from popping up. As long as we can get the antichains under control, the size of the branches will take care of itself.

4.5 Homework!

1. Prove that the Aronszajn tree that we constructed in week 1 was in fact a well-pruned Aronszajn tree.
2. Take another look at the lemma that we proved today. Our Proof By Picture in class was a little less rigorous than one might hope. Can you prove formally that the object that we built in this proof is an antichain?

5 Infinite Trees: Day 5

5.1 Attempt II: Let's Everbranch This Sucker!

Yesterday we showed that if we can construct an everbranching tree that has no uncountable antichains it will be a Suslin tree. Let's try to construct such a thing, and see what goes wrong. Shouldn't be too hard. For our base step, let's take $T_0 = \{t_0\}$. For $T_{\alpha+1}$, let's take $\{a_n, b_n : t_n \in T_\alpha\}$. So far so good. This gives us our everbranchiness. So let's assume that somehow in spite of this, we've failed to construct a Suslin tree.[19] The problem must be that we've accidentally created an uncountable antichain somewhere along the way.

Claim: If A is a maximal antichain in T, then for any $\alpha < \omega_1$, there exists an ordinal γ such that $\alpha < \gamma < \omega_1$, and $A \cap T^\gamma$ is a maximal antichain of T^γ.

That is, if A is a maximal antichain of T, then there were an uncountable number of moments in our construction of T at which A already popped up as a maximal antichain.

Proof. Clearly, for any ordinal γ, $A \cap T^\gamma$ is an antichain of T^γ. We want to find a γ so that A is maximal—that is, for any $t \in T^\gamma$, there exists $a_t \in A \cap T^\gamma$ such that a_t is comparable to t.

We know that this is true in the larger tree. So for each $t \in T$, define $a_t \in A$ to be a node in A which is comparable to t.

We will define an ω-sequence of ordinals $\beta_0 < \beta_1 < \beta_2 < \beta_3 < \dots$ by setting $\beta_0 = \alpha$, and

$$\beta_{n+1} = \bigcup_{t \in T^{\beta_n}} \mathrm{ht}(a_t)$$

for each $n \in \omega$. Notice that for each $n > 0$, we have $\{a_t : t \in T^{\beta_{n-1}}\} \subseteq T^{\beta_n}$.

Let $\gamma = \sup_{n\in\omega} \beta_n$, and consider T^γ. for any $t \in T^\gamma$, we have $t \in T^{\beta_n}$ for some $n \in \omega$. Therefore, $a_t \in T^{\beta_{n+1}} \subseteq T^\gamma$. Thus $A \cap T^\gamma$ is a maximal antichain in T^γ. $\square$

[19] This does, after all, appear to be the most likely result.

5.2 How To Kill An Antichain

How should we think about this claim that we've just proved? Well, if we have an uncountable antichain, the claim essentially says that we had an uncountable number of opportunities to kill it as we constructed our tree—we just sort of never got around to it. But killing antichains is pretty easy. Here's how it works:

If A is going to develop into a maximal antichain, then it is already maximal in T^γ for some limit ordinal γ. This means that for every $t \in T^\gamma$, there exists an element $a_t \in A$ such that either $a_t < t$ or $a_t > t$. Since T^γ is everbranching, we can take the partial branch from the root to $\max\{a_t, t\}$, and extend it to get a full branch of length γ. Call this branch B_t.

If we want to construct T_γ in such a way that A never becomes uncountable, we can simply let $T_\gamma = \{c_t : t \in T^\gamma\}$, with ordering given by $c_t > b$ for every $b \in B_t$.[20] Now every element in level T_γ or higher is comparable to some element of A, and so our antichain cannot grow to be uncountable.

Okay, so what's the problem, then? If we can kill off an antichain of our choice at every limit step, and there are $\aleph_1$ limit steps on the way to the top, surely that gives us enough freedom to make sure we don't run ourselves into trouble? Except... an antichain is just some subset of the tree T. We have $\aleph_1$ elements of T, and so we have $2^{\aleph_1}$ subsets. This gives our tree a lot of room to throw nastiness at us. If we knew enough about the future to be able to predict which antichains could become uncountable, we'd be in good shape. But we need more information than we've got. In fact, we need more information than we can get in ZFC.

[20] That is, c is the **cap** of the branch B_t.

5.3 Homework!

1. Review your notes and ask questions!
2. The **cofinality** of an ordinal number κ is the smallest ordinal α such that there exists a cofinal map $\alpha \to \kappa$. Find the cofinality of ω, $\omega + 1$, $\omega 2$, and ω_1.
3. An ordinal number is called a **cardinal number** if there is no injection $\phi : \kappa \to \alpha$ from κ into a smaller ordinal α.
   - Show that if $\mathrm{cof}(\kappa) = \kappa$, then κ is a cardinal number.
   - Is the converse true? Either show that every cardinal satisfies $\mathrm{cof}(\kappa) = \kappa$, or find a counterexample.

6 Infinite Trees Day 6

6.1 Definitions! Yaaaay Definitions!

A function $f : \alpha \to \kappa$ is said to be **cofinal** in κ if for any $\beta < \kappa$ there exists $\delta \in \alpha$ with $f(\delta) \geq \beta$.

The **cofinality** of an ordinal number κ is the smallest ordinal α such that there exists a cofinal map $f : \alpha \to \kappa$. For example, the cofinality of any successor ordinal is 1. The cofinality of ω is ω, as is the cofinality of $\omega 2$. The cofinality of ω_1 is ω_1. Essentially, the cofinality of an ordinal number κ tells you the smallest number of ordinals you need to line up to shoot to the top of κ.

We call an ordinal κ **regular** if $\mathrm{cof}(\kappa) = \kappa$. An ordinal will be regular whenever we can't find a smaller (in order type) collection of ordinals whose union is κ.

Let κ be a regular uncountable ordinal, and let $C \subseteq \kappa$. We call C **closed** if for any subset $S \subseteq C$ with $|S| < |\kappa|$, we have

$$\left(\bigcup_{\alpha \in S} \alpha \right) \in C.$$

That is, limits of sequences of ordinals in C are in C.

We call C **unbounded** if for any $\alpha \in \kappa$ there exists $\beta \in C$ with $\alpha < \beta$. If a set is **cl**osed and **unb**ounded, we call it a **club**.[21]

Example: κ is a club in κ.

Less Stupid Example: $\{\alpha \in \kappa : \alpha \text{ is a limit ordinal}\}$ is a club in κ

Kind of Awesome Example: $\{\beta \in \omega_1 : A \cap T^\beta \text{ is a maximal antichain in } T^\beta\}$ is a club in ω_1.

Let's prove some things about clubs!

Theorem: Let κ be regular and uncountable. Then for any $\gamma < \kappa$, and any collection $\{C_\alpha\}_{\alpha<\gamma}$ of clubs, their intersection $C = \bigcap_{\alpha<\gamma} C_\alpha$ is also a club.

Proof. For any $S \subseteq C$, we have $S \subseteq C_\alpha$ for any $\alpha < \gamma$. Then we have

$$\bigcap_{\beta \in S} \beta \in C_\alpha$$

[21] Get it? Because it's CLosed, and UnBounded! Logicians, am I right?

for any $\alpha < \gamma$, since each of the C_α is closed. It follows that C is closed.

Now we must show that C is unbounded. In order to do this, it will suffice to find a function $h : \kappa \to C$ such that $h(\beta) > \beta$ for every $\beta < \kappa$.

For any $\alpha < \gamma$, define $f_\alpha : \kappa \to \kappa$ such that $f_\alpha(\beta)$ is the least $\lambda \in C_\alpha$ with $\lambda > \beta$. Define $g : \kappa \to \kappa$ by

$$g(\beta) = \bigcup_{\alpha<\gamma} f_\alpha(\beta).$$

This is less than κ, since κ is regular. And $g(\beta) \geq f_\alpha(\beta) > \beta$ for every $\beta < \kappa$. However, g is probably not the function we're looking for—we have no reason to believe that $g(\beta)$ would be in any of the C_α. So we still have some work to do.

For any $n \in \omega$, define g^n to ge g composed with itself n times. We define

$$g^\omega(\beta) = \bigcup_{n\in\omega} g^n(\beta).$$

Clearly, $\beta < g^\omega(\beta) < \kappa$. Also, for any α, we have

$$f_\alpha(\beta) \leq g(\beta) \leq f_\alpha(g(\beta)) \leq g^2(\beta) \leq \dots$$

This means that for all α,

$$g^\omega(\beta) = \bigcup_{n\in\omega} f_\alpha(g^n(\beta)) \in C_\alpha.$$

Thus $g^\omega(\beta)$ is in C, and we have shown that C is unbounded. $\square$

6.2 Homework!

1. Prove that if $\mathrm{cof}(\kappa) = \kappa$, then κ is the smallest ordinal of its size. That is, $|\alpha| < |\kappa|$ for all $\alpha < \kappa$.
2. Prove that if κ is a regular uncountable ordinal and $f : \kappa^n \to \kappa$, then

$$C = \{\alpha < \kappa : \alpha \text{ is closed under } f\}$$

   is a club in κ.

7 Infinite Trees Day 7

7.1 Clubs In Trees

Let $T = (\omega_1, <_T)$ be a tree. That is, T has $\aleph_1$ nodes, and we've labeled all of them somehow with the elements of ω_1. Then

$$C = \{\alpha < \omega_1 \mid \alpha \text{ a limit ordinal and } T^\alpha = \alpha\}$$

is a club in ω_1. That is, the set of limit ordinals α where T^α just happens to contain exactly the elements of α is a club.

It seems bizarre that this should ever be the case, but it turns out that in a set-theoretic sense, this happens almost all the time. Let's prove it.

Proof. First we'll prove that C is closed. Let $S \subseteq C$, with $|S| < \aleph_1$. Let $\bigcup_{\alpha \in S} \alpha = \gamma$ Then we have $\bigcup_{\alpha \in S} T^\alpha = T^\gamma$. If $\beta < \gamma$, then $\beta < \alpha$ for some $\alpha \in S$. Thus $\beta \in T^\alpha \subseteq T^\gamma$, so we have $\gamma \subseteq T^\gamma$. Conversely, if $\beta \in T^\gamma$, then $\beta \in T^\alpha$ for some $\alpha \in S$, and so $\beta \in \alpha \subseteq \gamma$. Thus $T^\gamma \subseteq \gamma$. Therefore, $\gamma \in C$, and C is closed.

To show that C is unbounded, we choose an arbitrary $\beta < \omega_1$. We want to find $\alpha > \beta$ with $\alpha \in C$.

We define $\{\alpha_n\}_{n \in \omega}$ as follows: Take $\alpha_0 = \beta$. Let α_1 be the smallest ordinal greater than or equal to α_0 such that $T^{\alpha_0} \subseteq \alpha_1$. Let α_2 be the smallest ordinal greater than or equal to α_1 such that $\alpha_1 \subseteq T^{\alpha_2}$. We continue in this way until we have

$$T^{\alpha_0} \subseteq \alpha_1 \subseteq T^{\alpha_2} \subseteq \alpha_3 \subseteq T^{\alpha_4} \subseteq \dots$$

We set $\alpha = \bigcup_{n \in \omega} \alpha_n$. We want to show that $T^\alpha = \alpha$. Let $\delta < \alpha$. Then $\delta \in \alpha_n$ for some odd n. Then $\delta \in T^\alpha_{n+1} \subseteq T^\alpha$. Thus $\alpha \subseteq T^\alpha$.

If instead we suppose $\delta \in T^\alpha$, then we know that $\delta \in T^{\alpha_n}$ for some even n and so $\delta \in \alpha_{n+1} \subseteq \alpha$. Thus $T^\alpha \subseteq \alpha$, and we have $T^\alpha = \alpha$ and thus $\alpha \in C$.

□

Here's another club we're going to care about: Let $A \subseteq \omega_1$ be a maximal antichain of T. Then

$$D = \{\alpha < \omega_1 \mid A \cap T^\alpha \text{ is a maximal antichain of } T^\alpha\}$$

Let's show that this is a club.

Proof. Unboundedness is the tricky part here, but you may notice that we already proved that this collection of sets is unbounded.[22]

To show that D is closed, take some $S \subseteq D$ with $|S| < |\omega_1|$, and let $\bigcup_{\alpha \in S} \alpha = \gamma$. Let $x \in T^\gamma$. Then $x \in T^\alpha$ for some $\alpha \in S$. Thus there exists $a_x \in A \cap T^\alpha \subseteq A \cap T^\gamma$ such that x is comparable to a_x. This shows that $A \cap T^\gamma$ is a maximal antichain of T^γ, and thus D is closed. $\square$

7.2 Stationary Sets

Let κ be a regular uncountable ordinal. We call a subset $S \subseteq \kappa$ a **stationary set** if $S \cap C \neq \emptyset$ for any club $C \subseteq \kappa$.

Theorem: If S is stationary, and C is a club, then $S \cap C$ is stationary.

Proof. Let C_0 be a different club. Then $(S \cap C) \cap C_0 = S \cap (C \cap C_0)$. Since $C \cap C_0$ is a club, this intersection must be nonempty. $\square$

7.3 The Diamond Principle

Let's play a game. You choose a subset X of ω_1. The game has ω_1 turns, and on the α'th turn, I choose a subset G of α. If $G = X \cap \alpha$, then I win this round. Otherwise, you win the round.

We say that I **fail miserably** if I lose on a club. The game's a lot easier for you, so all I really want to do is avoid failing miserably. Thus it's my goal to win on a stationary set.

Fortunately, I can tell the future. Well, sort of. See, I have this sequence of guesses $\{G_\alpha \mid \alpha \in \omega_1\}$ that will be right on a stationary set *no matter what the original set X was*!!!

Such a collection of guesses is called a "diamond sequence". Clearly, there's no reason to believe that such a sequence of guesses should exist. However

[22] Mini-exercise: Find where!

as it turns out, there's no way to prove that it doesn't. The existence of a diamond sequence is completely consistent with ZFC. And it's this diamond sequence that will build our Suslin tree for us.

The Diamond Axiom: We call a sequence $\{A_\alpha : \alpha < \omega_1\}$ a **diamond sequence** if

(i) $A_\alpha \subseteq \alpha$ for all $\alpha \in \omega_1$.

(ii) For any $X \subseteq \omega_1$, the set $\{\alpha < \omega_1 : X \cap \alpha = A_\alpha\}$ is stationary.

The existence of such a sequence is independent from ZFC. So while there's no reason on earth to believe that such a sequence exists, there's really no reason to believe that it shouldn't. The diamond axiom states that such a sequence exists. We often refer to this axiom as $\diamondsuit$.

7.4 $\diamondsuit$ and CH

Theorem: $\diamondsuit$ implies CH.

Proof. Let $X \subseteq \omega \subseteq \omega_1$. Then the set

$$S = \{\alpha < \omega_1 : X \cap \alpha = A_\alpha\}$$

is stationary. Let $C = \{\alpha < \omega_1 : \alpha > \omega\}$. Then C is clearly a club in ω_1.[23] Since S is stationary and C is a club, S and C intersect. Thus $X = X \cap \alpha = A_\alpha$ for some $\omega < \alpha < \omega_1$. Thus any element of $\mathscr{P}(\omega)$ is an element of our diamond sequence. Since $|\{A_\alpha : \alpha \in \omega_1\}| = \aleph_1$, it must be the case that $|\mathscr{P}(\omega)| = \aleph_1$. $\square$

[23] Proof by duh.

7.5 Homework!

Show that the following statements are equivalent to $\diamondsuit$.

(i) There are $A_\alpha \subseteq \alpha \times \alpha$ for all $\alpha < \omega_1$ such that for all $A \subseteq \omega_1 \times \omega_1$,

$$\{\alpha < \omega_1 : A \cap (\alpha \times \alpha) = A_\alpha\}$$

is stationary.

(ii) There exist $f_\alpha : \alpha \to \alpha$ for $\alpha < \omega_1$ such that for each $f : \omega_1 \to \omega_1$ there exists α such that $f|_\alpha = f_\alpha$ and $\alpha > 0$.

(iii) There are $f_\alpha : \alpha \to \alpha$ for $\alpha < \omega_1$ such that for each $f : \omega_1 \to \omega_1$ the set $\{\alpha : f|_\alpha = f_\alpha\}$ is stationary.

8 Infinite Trees Day 8

8.1 The Last Piece Of The Puzzle

The diamond principle gives us exactly the tool that we need to construct a Suslin tree. Let A_α for $\alpha < \omega_1$ be our $\diamondsuit$-sequence. Suppose $T = (\omega_1, <_T)$ is an ever-branching tree of height ω_1, and that the following holds:

Let $\alpha < \omega_1$ be a limit ordinal such that $T_\alpha = \alpha$ and A_α is a maximal antichain of T^α. Then for every $\beta > \alpha$ and $x \in T_\beta$, there exists $y \in A_\alpha$ such that $y <_T x$.

Then T is a Suslin tree.

Proof. Let A be a maximal antichain of T. Then there exists a club E such that for any $\alpha \in E$,

(i) α is a limit ordinal

(ii) $T^\alpha = \alpha$

(iii) $A \cap T^\alpha$ is a maximal antichain of T^α.

We've proved that each of these properties forms a club, so we intersect them to get E.

Because $\{A_\alpha\}$ is a diamond sequence, the set

$$S = \{\alpha < \omega_1 \mid A \cap \alpha = A_\alpha\}$$

is stationary. This means that there exists some $\alpha \in E \cap S$. So for some α, we know that A_α is the restriction of A to T^α. We know that for every $\beta > \alpha$ and $x \in T_\beta$, there exists $y \in A_\alpha$ such that $y <_T x$. This means that A can't possibly have any elements on levels above α, so $A_\alpha = A$. We know A_α is countable, so every maximal antichain in T must be countable. $\square$

8.2 Constructing the Suslin Tree

Let $\{A_\alpha \mid \alpha < \omega_1\}$ be a $\diamondsuit$-sequence. Let $\{\lambda_\alpha \mid \alpha \in \omega\}$ be the increasing enumeration of the limit ordinals in ω_1. Let $\lambda_0 = 0$.

The zero through ω'th levels are the infinite binary tree. Then we recursively construct the levels T_α for $\alpha > \omega$ so that they satisfy

(a) $T_\alpha = \{\lambda_\alpha + n \mid n < \omega\}$

(b) For each $n < \omega$, we have $(\lambda_\alpha + n) < (\lambda_{\alpha+1} + 2n)$, and $(\lambda_\alpha + n) < (\lambda_{\alpha_1} + 2n + 1)$.

(c) If $\beta < \alpha$ and $x \in T_\beta$, then there exists $y \in T_\alpha$ such that $x <_T y$.

(d) If α is a limit ordinal and $T^\alpha = \alpha$ and A_α is a maximal antichain of T^α, then for all $x \in T_\alpha$, there exists $y \in A_\alpha$ such that $y <_T x$.

The first condition ensures that each one of the ω_1 nodes has a specific place to go. The second condition ensures that our tree will be everbranching. The third condition guarantees that our tree will be well-pruned. The fourth condition employs the diamond principle in just the way we discussed in the last section. The diamond sequence guesses possible antichains, and if we get a possible problem antichain, we kill it.

Now all we have to do is show that this recursion can actually go through, and we'll be done. Ready? Let's go!

Clearly, the base of our tree is fine. The second condition completely defines our successor case. So all we need to do is deal with the limit stages. Suppose α is a limit ordinal, and T^α has already been defined.

Possibility 1: Suppose $T^\alpha = \alpha$ and A_α is a maximal antichain of T^α. For each $x \in T^\alpha$, there exists $y \in A_\alpha$ such that x and y are comparable. Otherwise A_α would not be maximal.

Choose $z_0 \in T^\alpha$ such that x and y are both $\leq z_0$, and let $\gamma_0 = \mathrm{ht}(z_0)$. Choose an increasing sequence $\gamma_0 < \gamma_1 < \gamma_2 \ldots$ with limit α, and choose $z_n \in T_{\gamma_n}$ such that $z_n < z_{n+1}$.

Define $B(x) = \{y \in T^\alpha \mid \exists n, y < z_n\}$. Then $B(x)$ is a branch in T^α of height α. Now let $T_\alpha = \{x_n \mid n \in \omega\}$, and for each $t \in T^\alpha$, put $t < \lambda_\alpha + n$ if and only if $t \in B(x_n)$.

Then the conditions all hold for level T_α.

Possibility 2: Possibility 1 doesn't happen. Then do the same construction, and kill off a random antichain just for fun.

8.3 Homework!

Remember Aronszajn trees? How about that exercise about special trees? I thought it was a pretty exercise, and I'm fairly certain nobody did it...

1. An Aronszajn tree is **special** iff it is the union of a countable number of antichains. Can you construct a special Aronszajn tree?

2. We call a tree $\mathbb{Q}$**-embeddable** iff there exists a (not necessarily injective) function $f : T \to \mathbb{Q}$ such that $f(x) < f(y)$ whenever $x < y$. Show that an Aronszajn tree is special if and only if it is $\mathbb{Q}$-embeddable.